PHYSICS 300 – –FALL 2007 – – LAB # 2

Coupled Oscillators

Introduction

In this experiment you are going to observe the normal modes of oscillation of several different mechanical systems, first on the air tracks and then using some coupled pendula. The normal modes of motion of a system of coupled oscillators are 'stable' with respect to time. That is, if you start the masses of a system oscillating in one of the normal modes and observe it for some time, the motion will have constant characteristics as its amplitude decays because of the ever present friction forces. Also, the frequency of oscillation of all the masses in the system is the same, and each of the masses executes simple harmonic motion at this frequency. The frequency is called the natural frequency of the normal mode or an eigenfrequency of the system. There are certain relationships between the eigenfrequencies and the frequencies of the uncoupled oscillators that will be discussed later.

If a system is excited into oscillations that are a mixture of normal modes, then the motion will change character over a period of time. Perhaps one of the masses will decrease amplitude and pass its energy to another mass, which will later pass the energy back. Such motion is clearly different from simple harmonic motion and therefore does not constitute a normal mode, even though the oscillations are all at the same frequency. The exchange of energy generally occurs at a frequency that is quite different from the oscillation frequency. (Read this paragraph a second time.)

Theory

Begin your study with two equal mass gliders on the air track as shown in Fig. 1. If the masses are not equal, certain complications arise that can obscure the simplicity of the motion, so you should remove all extra weights, clips, etc. and be sure that you have two sails of equal mass that you can mount on these gliders for damping the motion by air friction. The two gliders are coupled together with a spring as shown in the diagram. This coupling is somewhat tighter than the coupling you will later use with the two pendula, but it is not perfectly rigid; with a rigid bar the system would have only one mode of oscillation because the gliders would not be able to move relative to one another.

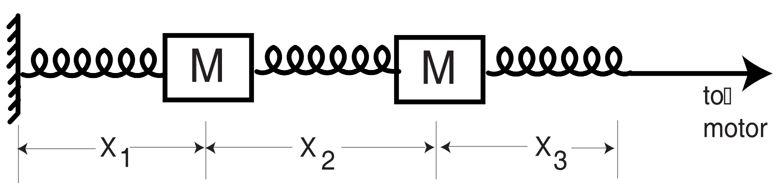

FIG. 1: The two masses and three spring constant are equal. At equilibrium, the three lengths x_i are also equal. The motor can only drive one end of one spring, but at resonance the energy is distributed among the masses.

If you displace the gliders symmetrically from their initial equilibrium positions and release them, their oscillations will die down in time, but the relative displacements of the two gliders will maintain a constant relationship. For example, if you push them toward one another by the same amount and release them, the displacement of one will always be equal in magnitude but opposite in direction to the displacement of the other. If you displace them both to one side by an equal amount and release them, they will oscillate together in a way that preserves their equilibrium separation. You should be able to observe both of these normal modes of oscillation with the pair of coupled masses before you go any further.

You should observe that the two normal modes described above have different frequencies. Measure these frequencies. You should rotate the sails on the gliders so that they are parallel to the air track to minimize damping in order to do this part of the experiment. There will then be many oscillations for you to count before all the energy of the system is lost. Now hold one of the masses fixed and measure the oscillation frequency of the other mass. Also, fix the second mass and measure the oscillation frequency of the first one. What is the relationship among the four frequencies you have measured?

In the case of one mass vibrating with the other held fixed the frequency is simply $\omega = \sqrt{2k/M}$ since the restoring force on the displaced mass is $2kx$ (because there are two springs each contributing kx). For the oscillations with the distance between the gliders constant, the frequency is $\omega = \sqrt{2k/2M}$ since there might just as well be a massless rigid bar between the gliders making it a mass of $2M$ oscillating under the same conditions as the single mass above. For the case of the masses oscillating in opposite directions, the frequency is $\omega = \sqrt{3k/M}$ since the restoring force from the middle spring is $2kx$ (its stretch is twice the displacement of either mass) plus the normal restoring force from the outside spring. The frequencies you measure should therefore be in the ratio of $1 : \sqrt{2} : \sqrt{3}$. Also notice that the sum of the squares of the individual oscillation frequencies (first one mass fixed and then the other) is equal to the sum of the squares of the frequencies of the normal modes. This is not an accident!

It is constructive to consider the motion of the center of mass (CM) of the system of gliders in each of the normal modes. In this case it is easy: in the high frequency mode (called the optical mode) the CM remains fixed at the middle of the center spring. You can hold the spring at this point without disturbing the oscillation. This point is called a node of the motion. In the low frequency normal mode (called acoustical mode) the CM oscillates at the acoustical frequency. In the optical mode there is motion with respect to the CM, but the CM is stationary. In the acoustical mode there is no motion with respect to the CM, but the CM moves. These are general properties of these two types of modes.

Consider the motion of the gliders with respect to the CM. How does the motion look from a coordinate system moving with the CM in each of the normal modes of oscillation? Describe it.

Procedure

The normal modes of motion represent the way a system of coupled masses oscillates naturally. In order to see what happens to a system of coupled oscillators driven at some frequency, consider the analogy to the case of a single mass where the motion is oscillatory at the driving frequency with an amplitude that is a maximum when the driving frequency is equal to the natural frequency. Since the coupled oscillators have two natural frequencies you might expect there to be two resonant maxima in the amplitude of the driven motion. In order to test this, you should mount the sails on the gliders for damping and proceed to drive them at various frequencies using the variable speed motor. You should record the amplitude of oscillation of each mass as a function of the driving frequency and plot the result. You should observe two resonant maxima for each mass as shown in Fig. 2, and the maxima should occur near the natural frequencies you have already measured.

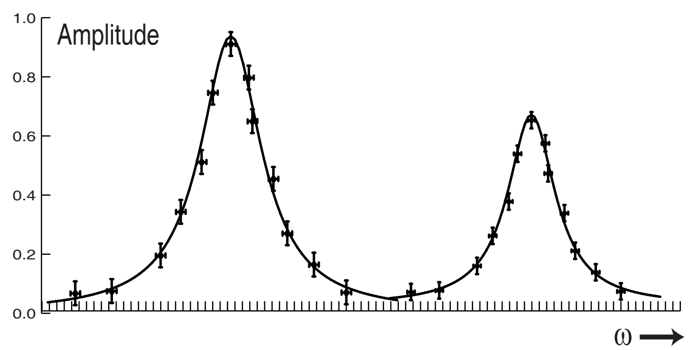

FIG. 2: The two resonances at the eigenfrequencies can be mapped out as shown. They are at different frequencies, and may have different heights and/or different widths.

Optional Advanced Work: If you would like to do some advanced work on this system you should go through exactly the same procedure for three coupled masses. There are now three normal modes. You might guess that one of them is an acoustical mode similar to that of two masses in which all three masses move together as if they were connected with rigid rods. This is a good guess but it's not correct simply because this middle mass would be expected to oscillate but the sum of the forces on it from the two springs would always be zero. This is because these springs retain their equilibrium length. It is therefore necessary that the middle mass oscillate with a larger amplitude but at approximately the same phase as the outside masses in the acoustical mode. This mode is hard to observe by starting the gliders oscillating at some point because you don't know the amplitude of the center glider. The frequency would be $\omega = \sqrt{2k/3M}$ as long as all three gliders have equal mass.

A little thought might lead you to guess that another normal mode consists of the two end masses oscillating in opposite directions with the center one perfectly stationary. You can verify this by starting the motion and then seeing if it maintains itself and if both the masses oscillate with the same frequency. Actually the center one can be said to oscillate at that frequency too, but with zero amplitude. If you do it carefully, the middle mass will never move as the oscillation of the end ones dies out. Show that the frequency of this oscillation is given by $\omega = \sqrt{2k/M}$.

The third normal mode is a little more difficult to discover by simply studying the equipment. You can find it by putting the sails on the gliders and driving the system at various frequencies. You can find the approximate frequency by using the rule of the sum of the squares of the frequencies described before. The oscillation frequency of each individual mass, if the the others were clamped, is $\omega = \sqrt{2k/M}$ (they're all equal) and so the sum of the squares is $6k/M$. The two frequencies we have discussed can be squared and summed to give $(8/3)k/M$ so the frequency of the third normal mode is expected to be $\omega = \sqrt{10k/3M}$.

This is higher than the other two frequencies, but if is still within the range of your motors. You should measure the oscillation frequencies of the clamped masses to verify the above calculation, and when you find the normal mode, you should verify the sum of the squares relation.

Plot the amplitude of oscillation of one of the masses versus frequency and you will find *three* resonance maxima, two of which correspond to the normal modes we have already predicted. Observe the first normal mode and measure the amplitude and phase of the center glider. Observe the third normal mode and measure its frequency. If you set the drive at the resonance frequency of any of the modes and then switch off the motor, will the oscillations decay without changing frequency and still maintain the characteristics of the motion? If you are careful, you should be able to observe this.

Consider the motion of the CM of the three coupled oscillators. Describe the motion of the CM for each of the three normal modes of oscillation. Describe the motion of each of the gliders with respect to the CM for each of the three normal modes. Could you do this for motion of the system which is a mixture of normal modes?

Hints and Kinks Department

You must remember to wait some time for oscillations at the undesirable frequencies to die out after you change the driving frequency of the variable speed motor. Otherwise you will have the kind of problems of mixed modes described at the end of Experiment 1, namely, the motion will be a superposition of two or more frequencies. With coupled oscillators the system is much more complicated and too difficult to analyze in that simple way, so it's important to be patient.

The damping time of an air track glider with sail is also considerably longer than the spring in Experiment 1. You may want to measure it by clamping one glider and just watching the other one's oscillations damp out (of course, with no drive). Suffice it to say that the motion at any frequency which is not a pure eigenfrequency will decay into a superposition of the normal modes of the system resulting in a rather complicated motion.

It is very important to match the masses of the system as closely as possible. If you don't, the normal modes will not be symmetrical in the coordinate displacements. Of course normal modes exist for any system, whether or not the masses are equal. Usually the displacements are inversely proportional to the masses so that the CM of the system behaves in the same way as the center-of-mass of a system with equal masses. Remember that the sails do not have zero mass, so that the measurements you make of the resonant frequencies with other masses clamped should have the sails mounted but turned parallel to the track.

Be sure you do not drive the system into oscillations that are so large that the springs pop off. Be careful not to overheat the motors.

Coupled Oscillators With Variable Coupling

Connect the strings of two pendula with a massless (almost) rigid rod such as a soda straw. You can do it by cutting short slits in the ends of the straw and slipping the string of each pendulum into the slits at the end of the straw. The distance between the pendulum strings should be just a few mm less than the length of the straw. You should slide the coupling rod up the string until 9/10 or more of the string is below the coupling. Find and describe the normal modes of oscillation in the plane of the strings of this system of coupled oscillators (hint: there is an acoustical and an optical mode). What are the frequencies? Show that the measured values of the frequencies are consistent with the expected values. Do they also satisfy the sum of the squares rule? (How would you 'clamp' one of the masses in this case?)

You should observe and measure the frequencies of the normal modes for several different heights of this coupling bar. Make a plot of the frequencies of the normal modes versus height of the coupling bar. Is the result what you expect? Why or why not? Prove that the frequency of one of the normal modes is independent of the height of the bar, and that the other normal frequency depends on the square root of the distance from the bar to the pendulum bob. Which is the optical mode and which is the acoustical?

Put the bar near the top of the strings, displace one of the pendulum bobs, and release it. What happens? Describe the motion in detail. Go back and read the second paragraph of this write-up for a third time. How long does the phenomenon you observe take? This time depends on the degree of coupling and therefore on the height of the bar. Measure the time it takes versus height of the bar and make a graph. Does the curve look familiar? Is it related to anything we have done before? Is it related to the eigenfrequencies? How? Plot the time for energy transfer versus eigenfrequency to find out. Speculate on the formula for the time versus bar height graph. Can you derive it?

If you put the bar too low (lower than about 1/2 of the length of the strings) you will find that there is substantial coupling to another normal mode of oscillation, the torsional mode. Be careful to avoid it in the measurements above, but if you choose to do advanced work on this experiment you should study this mode as well as other normal modes of oscillation of the system. There are a total of four normal modes - why is this? Can you observe them all? Describe them.

For further advanced study you should try coupling three pendula by putting a slit in the middle of the straw and slipping the string of a third pendulum into it (see Fig. 3). What measurements can you make with this system? What can you predict about the results? Can you describe the motion of and with respect to the CM? Can you do it for just two pendula? Can you verify your predictions about the motion?

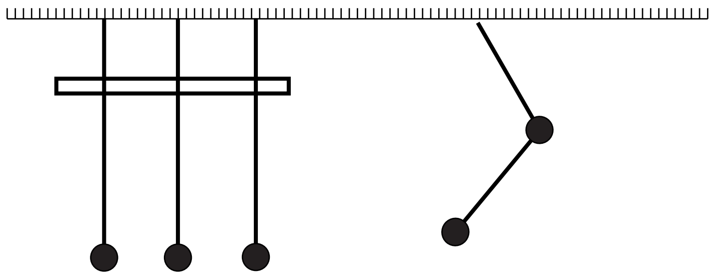

FIG. 3: The left side shows three coupled pendula, although you'll start this experiment using only two pendula. On the right is a "compound pendulum", which is simply another form of coupled oscillators.

You should also make a double pendulum, as shown on the right side of Fig. 3. What are the normal modes of motion? What are their frequencies? Make measurements. Do the frequencies obey the sum of the squares rule?

There are many options open to you in this experiment. Pick one or some of them that interest you and do them. Work carefully and accurately so that your efforts will not be wasted. It is better to do a small amount of work well than to try everything and do a sloppy job.

Hints and Kinks Department

Pick an amplitude that is large enough to be easily observable, but not so large that the pendulum weights collide. Remember also that a pendulum only executes simple harmonic motion if the maximum of excursion is a few degrees.

In both this and the air track experiments you should try to keep the oscillating masses from being disturbed by random air currents. Don't move around excessively, and don't breathe on the oscillators while you are trying to take measurements.