STATE UNIVERSITY OF NEW YORK AT STONY BROOK
DEPARTMENT OF PHYSICS AND ASTRONOMY
Part I.
Wednesday, 20 January 1999 – Day 2
Comprehensive Examination in Quantum Mechanics
and in Statistical Mechanics and Thermodynamics

General instructions: In each of the two areas, do two of the three problems. Each problem should take about $\frac{3}{4}$ hour and is worth twenty points. If a problem has subparts, each of these will be equally weighted, unless indicated otherwise, with the sum totaling twenty points. Use one examination book per problem and label it carefully with your name, the name of the problem's author, and the date. You may not use any materials other than this examination paper and the exam books supplied, a calculator, and, with the proctor's approval, a foreign language dictionary. None of these materials may be shared between students.

Quantum Mechanics

Three problems, work any two.

QM I. (Kuo)

A one-dimensional harmonic oscillator has a Hamiltonian

$$\mathcal{H}_H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

The lowering operator is defined as

$$a = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}} \left(x - \frac{i}{m\omega}p\right)$$

and the raising operator $a^\dagger$ is the Hermitean conjugate of a.

a. Calculate the commutator $[a, a^\dagger]$.

b. Calculate the commutator $[\mathcal{H}_H, a^\dagger]$.

c. The ground state $|0>$ obeys $<0|0>= 1$ and $a|0>= 0$. Evaluate $<0|x^2|0>$.

d. Find the values of $<0|p^2|0>$ and $<0|\mathcal{H}_H|0>$.

e. Prove that $<0|x^4|0>= 3<0|x^2|0>^2$.

f. Use these results to obtain a least upper bound on the ground state energy of the quartic oscillator with Hamiltonian

$$\mathcal{H}_Q = \frac{p^2}{2m} + \frac{1}{4}Kx^4.$$

You may consider the ground state $|0>$ of $\mathcal{H}_H$ as a trial state with variable ω.

QM II. (Lourie)

An electron interacts with a magnetic field $\vec{B}$ via the Hamiltonian

$$\mathcal{H} = -\vec{\mu} \cdot \vec{B}$$

with magnetic moment $\vec{\mu} = -e\hbar\vec{\sigma}/2mc$. Let $\vec{B} = B_y\hat{y} + B_z\hat{z}$ with $B_y \ll B_z$.

a. Calculate exact energy eigenvalues. Expand them in power series in B_y/B_z to leading order.

b. Treat B_y as a perturbation. Find the perturbed energies to second order in B_y/B_z. Compare to the result from (a).

QM III. (Sterman)

A beam of *electrically neutral* atoms, with mass m, spin 1/2 and magnetic moment μ, is moving in the x direction with momentum $p = \hbar k$. This quantum-mechanical wave is incident from $x < 0$ on a region of magnetic field,

$$\vec{B}(\vec{r}) = B\hat{z}\theta(x)\,,$$

with B a constant. $\theta(x)$ is the usual unit step function, zero for negative x and 1 for positive x.

a. (4 points) Suppose that the incident beam has spin +1/2 in the y direction. Construct the wave function for this state.

Hint: recall that in conventional notation, where

$$|\uparrow> = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

means a state with $\sigma_z = 1$:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

b. (10 points) Working to lowest order in the magnetic field, construct the wave function for the wave that is reflected at the boundary where the field begins. (Remember that the atoms are neutral.)

c. (6 points) What is the spin direction of the reflected wave?

Statistical Mechanics and Thermodynamics

Three problems, work any two.

SM&T I. (Shrock)

The 1D Ising model of a lattice of spins along the x-axis is defined by the Hamiltonian

$$\mathcal{H}_I = -\epsilon \sum_{<ij>} s_i s_j - B \sum_i s_i$$

over nearest neighbor pairs. The s_i are numbers, either +1 (spin up) or -1 (spin down); B is a magnetic field pointing along x. Assume that ϵ is positive (ferromagnetic). Set Boltzmann's constant $k_b = 1$ and define $\beta = 1/T$ and $K = \beta\epsilon$. In the following assume that the thermodynamic limit has been taken.

For the zero-field case, calculate (e.g., by summing a high-temperature series or via transfer matrix methods) the following for all $T > 0$:

a. spin-spin correlation function $G(n) =< s_0 s_n >$

b. correlation length ξ (defined by $G(x) \sim \exp(-x/\xi)$)

c. susceptibility

d. spontaneous magnetization

e. Describe the behavior of your answers for (b), (c), (d) as T goes to zero and also at $T = 0$.

SM&T II. (Verbaarschot)

Consider a gas of massless particles in a box of volume $V = L^3$.

a. Find the equation of state, i.e., the relation between pressure and the energy density using dimensional considerations.

b. Derive the same result from considering the momentum transfer at the boundary.

c. Photons are bosons. Can they condense? Why or why not?

SM&T III. (Shuryak)

$N = 10^6$ identical atoms kept in the same internal state with integer total angular momentum are placed in an isotropic 3-d harmonic trap, $V = m\omega^2 r^2/2$, with $\omega = 1$ kHz. Interaction between atoms can be ignored.

a. (5 points) For sufficiently high temperatures, find momentum and spatial distribution of particles and $< p^2 >, < x^2 >$.

b. (10 points) Derive and estimate numerically the critical temperature for Bose condensation T_c in this 3-d trap.

Hint: since N is large number, use semiclassical expressions or convert your sum into an integral.

c. (5 points) For sufficiently low temperatures, $T < T_c$, describe momentum and spatial distribution of particles.