STATE UNIVERSITY OF NEW YORK AT STONY BROOK
DEPARTMENT OF PHYSICS AND ASTRONOMY

Part I.

Wednesday, 1 September 1999 – Day 2

Comprehensive Examination in Quantum Mechanics and in Statistical Mechanics and Thermodynamics

General instructions: In each of the two areas, do two of the three problems. Each problem should take about $\frac{3}{4}$ hour and is worth twenty points. If a problem has subparts, each of these will be equally weighted, unless indicated otherwise, with the sum totaling twenty points. Use one examination book per problem and label it carefully with your name, the name of the problem's author, and the date. You may not use any materials other than this examination paper and the exam books supplied, a calculator, and, with the proctor's approval, a foreign language dictionary. None of these materials may be shared between students.

Quantum Mechanics

Three problems, work any two.

QM I. (Roček)

Consider a free particle moving in one dimension on an interval of length L.

a. For periodic boundary conditions $\psi(x) = \psi(x + L)$, find all the correctly normalized traveling wave eigenstates with their energies.

Now turn on a weak periodic potential $V(x) = 2V_0 cos(2\pi x/d)$ where $d = L/N$ for some large number N.

b. What are the nonvanishing matrix elements of $V(x)$ with respect to the unperturbed eigenstates?

c. Use this to find the change in the energy to second order in perturbation theory.

QM II. (Siegel)

Consider a particle of spin 1. Take the usual Cartesian basis for the spin components of the wave function, *i.e.*

$$\vec{\psi} = \begin{pmatrix} \psi_x \\ \psi_y \\ \psi_z \end{pmatrix}.$$

a. Consider the set of matrices S_i where the components $(S_i)_{jk}$ $(i = 1, 2, 3 \text{ or x, y, z})$ are given by

$$(S_i)_{jk} = -i\hbar\epsilon_{ijk}$$

or explicitly by

$$S_x = -i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad S_y = -i\hbar \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad S_z = -i\hbar \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Show that these matrices have the commutation relations of angular momentum so that $\vec{S}$ can be used to represent the spin of this particle.

b. Instead of the usual, assume the Hamiltonian

$$H = \alpha \vec{S} \cdot \vec{P}$$

in terms of the momentum operator $\vec{P}$, for some constant α. Write the time-dependent Schrödinger equation in position space for $\vec{\psi}$ in **3-vector notation** (**not** matrix notation: eliminate $\vec{S}$).

c. Take $\vec{\psi} = \vec{E} + i\vec{B}$ and show that $\vec{\psi}$ satisfies two of Maxwell's equations. Find the value of α.

d. Find and interpret the eigenstates with definite momentum in the $\hat{z}$ direction.

QM III. (Marx)

Consider a system with 2 energy eigenstates that are normalized and orthogonal:

$$\psi_1 \text{ with } E_1 = \hbar\omega_1 \qquad \psi_2 \text{ with } E_2 = \hbar\omega_2$$

$$<\psi_1|\psi_1> = <\psi_2|\psi_2> = 1 \qquad <\psi_1|\psi_2> = 0.$$

Now consider two other states of the system (not energy eigenstates) defined by

$$\varphi_1 = \tfrac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \qquad \varphi_2 = \tfrac{1}{\sqrt{2}}(\psi_1 - i\psi_2).$$

a. Show that φ_1 and φ_2 are also normalized and orthogonal.

Consider a state of the system defined at $t = 0$ as

$$\psi(0) = \tfrac{1}{\sqrt{2}}(\psi_1 + \psi_2).$$

b. Find the probabilities at $t = 0$ that ψ is in state φ_1 or φ_2.

c. Find the probabilities for being in state φ_1 or state φ_2 at a later time t.

Statistical Mechanics and Thermodynamics

Three problems, work any two.

SM&T I. (Likharev)

Two classical particles are placed on a ring with $N > 2$ similar, available sites. The particles cannot occupy the same site, and if they occupy a couple of neighboring sites, their repulsion increases the energy of the system by Δ.

- **a.** Find the average energy $<E>$ of the system at temperature T.
- **b.** Find the energy variance $(\Delta E)^2 = <E^2> - <E>^2$.
- **c.** Is there any value of N at which the r.m.s. fluctuation ΔE is smaller than $<E>$? If yes, find the corresponding temperature range.

SM&T II. (Hemmick)

A double helix molecule can be represented as a "zipper". Consider N sites on each helix, with each opposing pair of sites on the two helices connected by a link. Breaking a link costs energy ϵ. The molecule can be "unzipped" (i.e., the helices separated) from one side only.

- **a.** Calculate the partition function Z, the free energy F, and the probablity to break the molecule completely at temperature T.
- **b.** Find the mean energy $<E>$ and mean number of open links $<n>$.
- **c.** Consider $<n(T)>$ in the limits of small and large T and qualitatively explain the results.

SM&T III. (Shuryak)

Consider the growth of an ice layer on the top of a lake of water in the winter. Take the temperature of the water to be at the freezing point, $T_{water} = 273$ K. For the ice thickness $x(t)$ to grow with time t, the air temperature must be lower, $T_{air} < 273$ K.

a. Derive an expression for the time-dependence $x(t)$ of the growth in thickness of the ice. Assume the usual linear heat trasfer mechanism, i.e., that thermal energy flow rate is equal to $k \cdot$*(area)* $\cdot$*(temperature gradient)*, where k is a constant for the ice.

b. In the following use $k_{ice} = 1.6$ J s^{-1} m^{-1} K^{-1}, $L_f = 334$ kJ/kg for the heat of freezing, and $\rho_{ice} = 920$ kg/m^3 for the density of ice. Take the air above the ice to be at $T_{air} = 263$ K. Estimate how long it takes

  i. for the ice to grow thick enough for walking, x$\approx$ 2.5 cm

  ii. for the ice thickness to grow to equal the depth of the lake, which we take to be 10 m.