STATE UNIVERSITY OF NEW YORK AT STONY BROOK
DEPARTMENT OF PHYSICS AND ASTRONOMY

Part I.

Wednesday, 24 January 2001 – Day 2

Comprehensive Examination in Quantum Mechanics and in Statistical Mechanics and Thermodynamics

General instructions: In each of the two areas, do two of the three problems. Each problem should take about $\frac{3}{4}$ hour and is worth twenty points. If a problem has subparts, each of these will be equally weighted, unless indicated otherwise, with the sum totaling twenty points. Use one examination book per problem and label it carefully with your name, the name of the problem's author, and the date. You may not use any materials other than this examination paper and the exam books supplied, a calculator, and, with the proctor's approval, a foreign language dictionary. None of these materials may be shared between students.

Quantum Mechanics

Three problems, work any two.

QM I. (Friedman)

A two-state system obeys the Hamiltonian $H = \epsilon\sigma_z + \Delta\sigma_x$, where the σ's are the Pauli matrices, and ϵ and Δ are real c-numbers.

a. Find the eigenvalues and eigenvectors for this Hamiltonian.

b. If at $t = 0$ the system is in the "up" eigenstate of σ_z (with the eigenvalue $+1$), calculate the probability that at a later time the system will be found in the "down" eigenstate of σ_z (with the eigenvalue -1).

QM II. (Goldman)

A particle of mass m moves in the one-dimensional potential $U(x) = -\alpha\delta(x)$, where $\alpha > 0$.

a. (8 points) Find the eigenvalue and the eigenfunction of the ground state.

b. (8 points) The system is initially in the ground state; then the strength of the potential is changed suddenly: $\alpha \to \alpha'$. What is the probability that the particle remains in the ground state?

c. (4 points) State explicitly the condition that allows one to treat the potential change in **b.** as "sudden".

QM III. (Likharev)

The so-called coherent (or "Glauber") state $|\alpha\rangle$ of the 1D harmonic oscillator may be represented as a specific linear superposition of stationary ("Fock") states $|n\rangle$:

$$|\alpha\rangle = \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} \frac{\alpha^n}{(n!)^{1/2}} |n\rangle, \tag{1}$$

where α is a (generally, complex) scalar c-number characterizing the state.

a. Prove that $|\alpha\rangle$ is an eigenstate of the "annihilation" operator

$$a = \sqrt{\frac{m\omega}{2\hbar}}(x + i\frac{p}{m\omega}). \tag{2}$$

b. Now, let the oscillator that initially was in the coherent state (Eq. 1) be affected by a short but arbitrary pulse of classical force $F(t)$:

$$F(t) \neq 0 \text{ at } 0 < t < \tau, \text{ with } \omega\tau \ll 1. \tag{3}$$

Prove that the resulting state of the oscillator is also a coherent state.

c. Find the new value of parameter α at $t > \tau$.

Statistical Mechanics and Thermodynamics

Three problems, work any two.

SM&T I. (Aleiner)

A Boltzmann gas with number of particles $N_0 \gg 1$ and temperature T_0 is contained in a spherical vessel of volume V. At $t = 0$, a small orifice (i.e., circular opening) with radius $a \ll V^{1/3}$ opens in the vessel. The particles in the vessel can then escape to the vacuum, but the characteristic escape time is much longer that the time of energy and momentum relaxation. Assume $E(t) = \frac{3}{2} k_B T(t) N(t)$.

a. (10 points) Calculate the rate of change of the particle number in the vessel, $dN(t)/dt$, and the rate of change of the energy in the vessel, $dE(t)/dt$. (Use of dimensional analysis may help here.)

b. (10 points) Solve the resulting differential equations and find the time dependences, $T(t)$ and $N(t)$.

SM&T II. (Kakushadze)

Consider the following bosonic system. Let $a_n^\dagger$ be the creation operators and a_n be the corresponding annihilation operators ($n = 1, 2, \ldots$) with the following commutation relations

$$\left[a_n^\dagger \ , \ a_m^\dagger\right] = 0 \ , \qquad [a_n \ , \ a_m] = 0 \ , \qquad \left[a_n , a_m^\dagger\right] = n\delta_{n,m} \ .$$

The Hamiltonian is

$$H = \sum_{n=1}^{\infty} a_n^\dagger a_n \ .$$

(This system arises upon quantization of a free boson on a cylinder, except that we ignore the zero mode for simplicity.) Note that the states of the form

$$|k_1, k_2, \ldots\rangle = (a_1^\dagger)^{k_1} (a_2^\dagger)^{k_2} \cdots |\text{vac}\rangle \ ,$$

where $k_1, k_2, \ldots$ are the occupation numbers taking non-negative integer values, while $|\text{vac}\rangle$ is the ground state annihilated by all a_n's, give a complete basis for the Hilbert space.

a. (7 points) Show that the partition function of this system is given by

$$\mathcal{Z}(\beta) = \prod_{n=1}^{\infty} (1 - q^n)^{-1} \ ,$$

where $q \equiv \exp(-\beta)$, and $\beta \equiv 1/T$ is the inverse temperature.

b. (7 points) Compute the free energy, F, entropy, S, and specific heat,

$$C = T \frac{\partial S}{\partial T} \ .$$

c. (6 points) Evaluate the high temperature ($T \gg 1$) behavior of the specific heat by taking the high T limit in each term in the corresponding sum. Sketch C as a function of T, and compare your answer with that expected from a classical treatment of the problem.

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Find the efficiency of "circle cycle" given by the equation $\frac{1}{P_0^2}(P-P_c)^2+\frac{1}{V_0^2}(V-V_c)^2=1$. Treat the working medium as an ideal gas with constant $\gamma=c_p/c_v=5/3$. To simplify calculations consider a "small" cycle, i.e., use $p=P_0/P_c\ll 1$ and $v=V_0/V_c\ll 1$. It is convenient to parameterize the cycle by parameter $t\in[0,2\pi]$: $P=P_c-P_0\cos t$, $V=V_c-V_0\sin t$.

a. (4 points) Find the total work during the cycle.

b. (8 points) Show that the heat transfer vanishes at two points of the cycle: $t=t_0,t_0+\pi$, where $t_0=\tan^{-1}\frac{\gamma v}{p}$.

c. (5 points) Find the total amount of heat received from the heater. Calculate the efficiency.

d. (3 points) Find the maximal and minimal temperatures during the cycle. Calculate the corresponding Carnot efficiency and compare it with the efficiency of the circle cycle you just found for the particular case $p=v\ll 1$.