STATE UNIVERSITY OF NEW YORK AT STONY BROOK
DEPARTMENT OF PHYSICS AND ASTRONOMY
Part I.
Tuesday, 28 August 2001 – Day 2
Comprehensive Examination in Quantum Mechanics and in Statistical Mechanics and Thermodynamics

General instructions: In each of the two areas, do two of the three problems. Each problem is designed to take about $\frac{3}{4}$ hour and is worth twenty points. You may take up to four and $\frac{1}{2}$ hours to work the problems, so you have more than enough time if you understand what each problem you are answering is about. If a problem has subparts, each of these will be equally weighted, unless indicated otherwise, with the sum totaling twenty points. Use one examination book per problem and label it carefully with your name, the name of the problem's author, and the date. You are allowed to have with you one 8-1/2″ by 11″ page of notes at the exam. It must be readable without magnification, and it will be examined by the exam proctor. The ONLY materials you may have are the examination paper, the exam "bluebooks", your single page of notes, a calculator, and a foreign language dictionary, which will also be examined by the proctor. You may not share any these materials with other students. **No more than four problems will be graded.** You must turn in all bluebooks in which you write anything. For any bluebooks that do not contain solutions to the four problems you want graded, **you must write on the covers of the covers of those bluebooks DO NOT GRADE.**

Quantum Mechanics

Three problems, work any two.

QM I. (Brown)

The classical Hamiltonian for a particle of mass m and charge e in an electromagnetic field is

$$H = \frac{1}{2m}\left[\vec{p} - \frac{e}{c}\vec{A}(\vec{r},t)\right]^2 + e\phi(\vec{r},t)\,,$$

where $\vec{A}$ and ϕ are the vector and scalar potential, respectively.

a. Write down the corresponding Schrödinger equation.

A gauge transformation introduces

$$\begin{aligned}\vec{A}(\vec{r},t) &\rightarrow \vec{A}'(\vec{r},t) = \vec{A}(\vec{r},t) + \nabla\chi(\vec{r},t)\\ \phi(\vec{r},t) &\rightarrow \phi'(\vec{r},t) = \phi(\vec{r},t) - \frac{1}{c}\frac{\partial\chi}{\partial t}(\vec{r},t).\end{aligned}$$

b. Show that the Schrödinger equation is invariant under such a change.

QM II. (Smith)

A quantum-mechanical system has three states: $|u_1\rangle$, $|u_2\rangle$, $|u_3\rangle$. Consider two operators A and B defined by:

$$A|u_1\rangle = |u_3\rangle\,, \quad A|u_2\rangle = |u_2\rangle\,, \quad A|u_3\rangle = |u_1\rangle\,; \qquad B|u_1\rangle = |u_1\rangle\,, \quad B|u_2\rangle = 0\,, \quad B|u_3\rangle = -|u_3\rangle\,.$$

a. (3 points) Write down the matrix representations of the operators A, A^2, B, B^2 in the u-basis. Which of these operators are observables?

b. (2 points) Show that A and B^2 are commuting operators.

c. (5 points) Construct a basis of common eigenvectors of A and B^2.

d. (10 points) Give the most general matrix form of an operator M which commutes with B. Repeat this for B^2 and A^2.

QM III. (Abanov)

Two "quantum beads" of mass m and charge e each can easily (no friction) move along a round "necklace" having the radius R. The beads interact with each other with a Coulomb interaction $V = \frac{e^2}{r}$, where r is the distance between them.

a. (4 points) Write the Schrödinger equation for the system in terms of the angular coordinates ϕ_1 and ϕ_2 of the beads, respectively (see figure).

b. (2 points) Rewrite the Schrödinger equation in the new variables: "center of mass" $\Phi = \frac{\phi_1+\phi_2}{2}$ and relative angle $\phi = \phi_1 - \phi_2$. Show that in these variables the motion is "separated" into a center of mass motion and a relative motion of the beads.

c. (4 points) Describe qualitatively the nature of the eigenstates corresponding to the low energy levels assuming that Coulomb interaction is very strong.

d. (7 points) Derive an expression for the low quantum-mechanical energy levels of the system assuming that Coulomb interaction is very strong .

e. (3 points) Formulate quantitatively the condition on the radius of the necklace that justifies use of the assumption of strong Coulomb interaction in **c.** and **d.**

leave room for figure

Statistical Mechanics and Thermodynamics

Three problems, work any two.

SM&T I. (Zahed)

The classical Heisenberg model of a magnet is defined by the Hamiltonian

$$H = -J\Sigma_{ij}\vec{S}_i\vec{S}_j - \Sigma_i\vec{S}_i\vec{H}\,,$$

where $\vec{S}_i$ is a 3-component unit vector, $\vec{S}_i^2 = 1$, placed on the sites of a lattice. "Classical" in the name of the model means that this vector is not discrete-valued; it can have any direction on the unit sphere. The first sum is over all nearest-neighbor links on the lattice, and their number for each site (called the coordination number) is c.

- **a.** (6 points) The mean field approximation is based on the idea that for each spin the sum of its neighbors can be approximated using the *average* magnetization $\vec{M} = < \vec{S}_i >$, and the correlations between spins can be ignored. Calculate the partition function Z in this approximation.
- **b.** (4 points) Using this Z, calculate the magnetization $\vec{M}$ by differentiating over the magnetic field $\vec{H}$; this leads to a non-linear equation for $\vec{M}$. (Assume that the material is isotropic.)
- **c.** (6 points) Expand the nonlinear part of the equation at small $\vec{M}$, rewriting it in the form

  $$\vec{H} = \vec{M}\left[a(T,J,c) + b(T,J,c)M^2\right].$$

  Identify the coefficients $a(T,J,c)$ and $b(T,J,c)$.
- **d.** (4 points) For $\vec{H} = 0$ find the critical temperature T_c above which only the trivial solution $\vec{M} = 0$ is possible. Find $\vec{M}(T)$ for T below this T_c.

SM&T II. (Drees)

A Quark-Gluon Plasma (QGP) is matter formed of gluons, quarks and anti-quarks. Assume all particles in the QGP are massless and that the plasma has vacuum quantum numbers: it is charge neutral, color neutral, and has equal number of quarks and anti-quarks, etc. The pressure of the QGP at a temperature T may be approximated by $P(T) = AT^4 - B$, where A, B are constants.

- **a.** (4 points) Calculate the entropy $S(T)$ and energy density $\epsilon(T)$ using thermodynamical relations.
- **b.** (8 points) For gluons (bosons with 8 color configurations and 2 polarizations) and 3 flavors of quarks and anti-quarks (fermions with 3 colors and spin 1/2) the constant A is given by the degrees of freedom for gluons, g_g, quarks, g_q, and anti-quarks, $g_{\bar{q}}$, as

  $$A = \left(g_g + \frac{7}{8}(g_q + g_{\bar{q}})\right)\frac{\pi^2}{90}\,.$$

  Explain how you can derive the expression for $P(T)$; assume B is an external pressure. Start with a classical approximation for energy density, compare to the exact result, and then evaluate quantum corrections by expanding Bose and Fermi distributions in powers of the exponent. Compare the result you obtain for A to the precise result given above.
- **c.** (4 points) For $B = 500$ MeV/fm^3 evaluate the temperature T_0 (in MeV) at which the QGP is in mechanical equilibrium with the pressure-less vacuum.
- **d.** (4 points) How do temperature, gluon and quark number change in an adiabatically expanding QGP, as a function of its volume V?

SM&T III. (McCoy)

An ideal Carnot refrigerator has four parts to its operating cycle; two are isothermal, and two are adiabatic. Consider making a mass m of ice at zero degrees from water with such a refrigerator in a room of temperature 20 degrees centigrade in the following way. Initially both the water and refrigerator are at room temperature, and the refrigerator quasistatically and reversibly removes heat from the water until it freezes.

a. How much work dW is required to remove an amount of heat dQ at temperature T?

b. How much total energy is needed is needed to make the mass m of ice?

Denote by C the specific heat and by L the latent heat of water.