STATE UNIVERSITY OF NEW YORK AT STONY BROOK
DEPARTMENT OF PHYSICS AND ASTRONOMY
Part I.
Tuesday, 22 January 2002 – Day 2
Comprehensive Examination in Quantum Mechanics and in Statistical Mechanics and Thermodynamics

General instructions: In each of the two areas, do two of the three problems. Each problem should take about $\frac{3}{4}$ hour and is worth twenty points. If a problem has subparts, each of these will be equally weighted, unless indicated otherwise, with the sum totaling twenty points. Use one examination book per problem and label it carefully with your name, the name of the problem's author, and the date. You may not use any materials other than this examination paper and the exam books supplied, a calculator, your one page help sheet, and, with the proctor's approval, a foreign language dictionary. None of these materials may be shared between students.

Quantum Mechanics

Three problems, work any two.

QM I. (Goldman)

A particle of mass m moves in a one-dimensional square well potential $V(x) = 0$ for $0 < x < a$, $V(x) = \infty$ outside. At time $t = 0$ its initial wave function is an even mixture of the first two eigenstates: $\Psi(x, t = 0) = A[\psi_1(x) + \psi_2(x)]$, where A is the normalization constant.

a. (6 points) Find $\Psi(x, t)$, the (normalized) wave function at time $t > 0$, and the probability density $|\Psi(x, t)|^2$.

b. (10 points) Calculate the expectation values of the position $\langle x \rangle$ and the momentum $\langle p \rangle$. Notice the oscillation in the time dependence. What is the angular frequency Ω of the oscillation?

c. (4 points) Calculate the expectation value of the energy $E = \langle H \rangle$ (Hint: there is an easy way). Compare the energy E with E_1 and E_2, the eigen-energies corresponding to ψ_1 and ψ_2.

You may need:

$$\int xe^{ax}\,dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right) \tag{1}$$

$$\int x\sin^2 x\,dx = \frac{x^2}{4} - \frac{x\sin 2x}{4} - \frac{\cos 2x}{8} \tag{2}$$

QM II. (Rijssenbeek)

Consider a symmetric, one-dimensional, potential of form:

$$U(x) = \frac{1}{2}k(|x| - x)^2 \tag{3}$$

with $k = 5.25$ keV/nm^2, and $x_0 = 0.150$ nm. In this well a single electron lives (mass $m_e = 511$ keV/c^2). Given is also that $\hbar c = 197.3$ eV×nm.

a. (3 points) Sketch the potential and indicate the scale on the axes.

b. (3 points) Calculate the energy level spectrum assuming that the offset $x_0 = 0$.

c. (10 points) For the potential of part a), the "bump" in the center leads to the possibility of tunneling and affects the unperturbed energy levels calculated in part b). Use the Wentzel-Kramers-Brillouin (WKB) approximation to write down an expression for the shift in the energy level ΔE_n due to tunneling to a degenerate state with unperturbed energy E_n.

d. (4 points) Taking the approximation $E_n << U(0)$, calculate the energy shift numerically.

QM III. (van Nieuwenhuizen)

Consider a particle on the x-axis with potential $U(x)$ such that $U(x)$ vanishes as $x \to \pm\infty$, and $U(x)$ is everywhere negative and nonsingular. Recall that the ground state for such a system is always a nondegenerate bound state.

a. (*4 points*) Define $V(x) = U(x) - E_0$ where E_0 is the ground state energy. Write the Hamiltonian in factorized form as $H = A^\dagger A + E_0$ where $A = c\frac{d}{dx} + W(x)$ and c is a constant. Determine c and $W(x)$. (Hint: express $V(x)$ in terms of the ground state wave function $\phi_0(x)$ and try the logarithmic derivative of $\phi_0(x)$ for W.)

b. (*6 points*) Consider now two systems, one with Hamiltonian $H_1 = A^\dagger A + E_0$ and another with Hamiltonian $H_2 = AA^\dagger + E_0$. Let $A^\dagger A = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1$ and $AA^\dagger = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_2$. Construct the $W_1(x)$ which gives $V_1 = a^2$, where a is a constant, and construct the corresponding $V_2(x)$. (Hint: the solution of the Riccati equation $\frac{d}{dx}y + y^2 = 1$ is given by $y(x) = \tanh x$).

c. (*4 points*) Show that A annihilates $\phi_0(x)$. Show that $H_1 = A^\dagger A + E_0$ and $H_2 = AA^\dagger + E_0$ have the same non-vanishing eigenvalues. Draw a picture of the eigenvalues of H_1 and H_2, both the discrete and the continuous ones. (Hint: act with A on H_1).

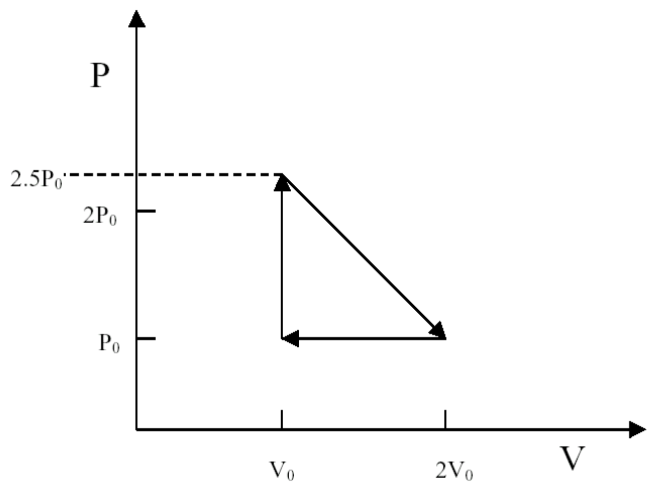
d. (*6 points*) If $A^\dagger A$ has a constant potential $V_1 = a^2$ the solutions for H_1 are plane waves. Prove that then the potential $V_2(x)$ of H_2 is also reflectionless. (A potential is called reflectionless if every incoming plane wave of the continuous spectrum is transmitted without reflection. In other words, there is total transmission).

Statistical Mechanics and Thermodynamics

Three problems, work any two.

SM&T I. (Hemmick)

An ideal monatomic gas is taken through the cycle shown in the figure below. The diagonal process is specially chosen so that initially heat flows into the gas, but later it reverses and flows out of the gas.



a. *(4 points)* Explain why the ratio of specific heats, γ, is $\frac{5}{3}$ for a monatomic gas.

b. *(8 points)* Find the pressure and volume at which the heat flow changes direction. (HINT: How does the diagonal compare to an adiabat at this point?)

c. *(8 points)* Determine the efficiency of this engine. (NOTE: If you cannot solve part b, pretend that the heat flow along the diagonal is always directed into the gas. State whether and why your calculation overestimates or underestimates the efficiency.)

SM&T II. (Schaefer)

A large number N of non-interacting, identical, spin-0 bosons of mass m occupies a volume V. The system is in contact with a heat bath at temperature T.

a. Compute the critical temperature T_c for Bose-Einstein condensation. Use

$$\int dx \frac{\sqrt{x}}{e^x - 1} = \Gamma(3/2)\zeta(3/2)$$

b. Show that there is no Bose-Einstein condensation if the particle motion is restricted to a plane (2 dimensions).

SM&T III. (van Neuwenhuizen)

Consider a small cubic cavity with volume v which is connected to a much larger cavity with volume V by a small opening. Both cavities are at temperature T and in equilibrium with each other.

a. (*4 points*) What is the number of states available to photons in the small cavity with frequencies between ν and $\nu + d\nu$? Consider wavelengths which are much smaller than the size of v.

b. (*4 points*) What is the average number n_A of particles in a given state A with energy E_A in the small cavity if the particles are (a) photons, (b) ^{4}He atoms, (c) ^{3}He atoms and (d) particles satisfying Boltzmann statistics? Denote the chemical potential by μ and the fugacity $e^{\mu/kT}$ by z. Do not calculate z but indicate how it can be determined if the total number of particles in the sum of v and V for the cases (b), (c) and (d) is N.

c. (*4 points*) Derive Planck's radiation law for the equilibrium energy density of electromagnetic radiation per unit volume in the frequency range $(\nu, \nu + d\nu)$. Denote the average energy of an oscillator with frequence ν at temperature T by $E(\nu, T)$. Your derivation may use the results in (1) and (2), and should take no more than 3 lines.

d. (*8 points*) Derive the following formula for the mean square of the fluctuations in the energy of the electromagnetic field in the small cavity for the frequencies between ν and $\nu + d\nu$
$< E^2 > - < E >^2 = [h\nu E(\nu, T) + E(\nu, T)^2] \frac{8\pi}{c^3} \nu^2 d\nu \; v$
(Hint: differentiate $E(\nu, T) = \Sigma E_n y^n / \Sigma y^n$ w.r.t. $\frac{1}{kT}$, where $E_n = (n + \frac{1}{2})h\nu$ and y^n are the Boltzmann factors.)

(Historical note: Einstein derived the equation in **d** in 1909 using Planck's radiation law as input. In 1925, Born, Heisenberg and Jordan were able to obtain the same result from quantum field theory without using this input).