STONY BROOK UNIVERSITY
DEPARTMENT OF PHYSICS AND ASTRONOMY
Part I.
Wednesday, 4 September – Day 2
Comprehensive Examination in Quantum Mechanics and in Statistical Mechanics and Thermodynamics

General instructions: In each of the two areas, do two of the three problems. Each problem should take about $\frac{3}{4}$ hour and is worth twenty points. If a problem has subparts, each of these will be equally weighted, unless indicated otherwise, with the sum totaling twenty points. Use one examination book per problem and label it carefully with your name, the name of the problem's author, and the date. You may not use any materials other than this examination paper and the exam books supplied, a calculator, your one page help sheet, and, with the proctor's approval, a foreign language dictionary. None of these materials may be shared between students.

Quantum Mechanics

Three problems, work any two.

QM I. (Aleiner)

The Hamiltonian of a harmonic oscillator with time dependent frequency has the form

$$\hat{H}(t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega_0^2 x^2}{2}\left(1 + \frac{1}{\cosh^2 \lambda t}\right).$$

At $t \to -\infty$ the system was in its ground state $|0\rangle$. Let us denote the probability for the oscillator to be excited into the state $|n\rangle$ as $t \to +\infty$ is $P_{0\to n}$.

- **a.** (6 points) Find $P_{0\to 1}$ for arbitrary value of λ.
- **b.** (7 points) Find $P_{0\to 2}$ for $\lambda \gg \omega_0$. Justify the use of the first order time-dependent perturbation theory.
- **c.** (7 points) Discuss the behavior of $P_{0\to 2}$ when $\lambda/\omega_0 \to 0$.

Hint: Use the fact that the only nonzero matrix elements of the coordinate operator x for a harmonic oscillator of frequency ω are $\langle n|x|n-1\rangle = \langle n-1|x|n\rangle = \sqrt{\frac{n\hbar}{2m\omega}}$.

QM II. (Verbaarschot)

Consider the following Hamiltonian describing two interacting spins:

$$H = \begin{pmatrix} 0 & \sigma_k A_k \\ \sigma_k A_k & 0 \end{pmatrix}$$

where the σ_k, $(k = 1, 2, 3)$ are the Pauli matrices, and the A_k are real valued constant fields. Below, you are asked to find certain matrices that are independent of A_k.

a. This Hamiltonian has an anti-unitary symmetry. Find it. What does this symmetry imply about the spectrum (wave functions and eigenvalues)?

b. This Hamiltonian also has a unitary symmetry. Find it.

c. In addition to the above symmetries, there is a real Hermitian operator that anti-commutes with the Hamiltonian. Find this matrix.

d. Based on these symmetries, what can you say about the eigenvalues of this Hamiltonian and the relations between its eigenvalues.

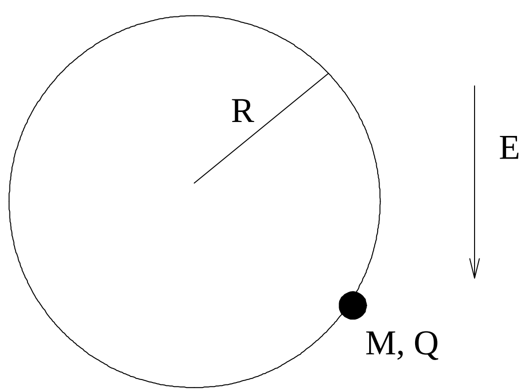
e. Verify your results for the spectrum by an explicit diagonalization of H.

QM III. (Hemmick)

Consider the figure below. A particle mass M and charge Q, is contrained to move around a frictionless circular loop. An electric field is present and directed "downwards" as indicated by the arrow in the figure. We will consider the motion of the particle in two limits:

a. (8 points) The field is strong. Calculate the energies of the lowest few states. (Hint: Consider the shape of the motion for a classical system).

b. (12 points) The field is weak enough to be considered as a perturbation to the otherwise free motion of the mass. Calculate the wave function and energy spectrum in the absence of the field. Calculate the change in the energy spectrum due to the electric field to the lowest *non-vanishing* order.

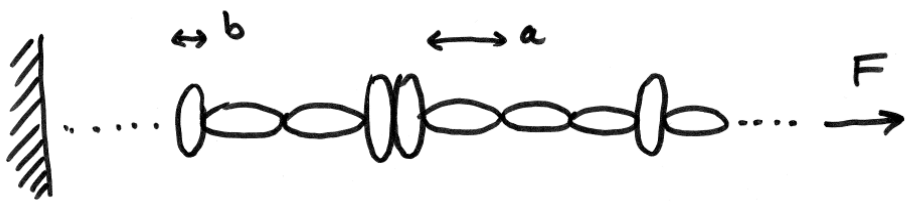


Statistical Mechanics and Thermodynamics

Three problems, work any two.

SM&T I. (Mihaly)

A rubber band can be modeled by a one-dimensional chain of linked segments, as shown in the Figure. The segments can not come apart. Any given element has two states: the "long" state, contributing distance a to the total length of the band, and "short" one contributing distance b. The chain is in contact with a thermal reservoir at temperature T. The only thermal motion of the elements is flipping between the long and short states. Let us denote the number of segments in the long and short states by N_l and N_s, respectively. The total number of segments, $N = N_l + N_s$ is constant.



- **a.** (2 points) What is the total length of the chain, if there is no energy difference between the long and short states?
- **b.** (2 points) One end of the chain is fixed, and the other end is pulled by an external force F. What is the length L of the chain in the limit of $T = 0$? What is the length if the temperature is very high? What is the sign of the thermal expansion coefficient? (Justify your answers with short statements; no calculation is necessary.)
- **c.** (6 points) We characterize the state of the system by a single parameter, $x = (N_l - N_s)/N$. For a given value of x, determine the entropy of the chain.
- **d.** (6 points) Determine the length of the chain at arbitrary temperature. Discuss the low and high temperature limits.
- **e.** (4 points) In what temperature range is Hooke's law ($F = \kappa \Delta L/L$, where ΔL is the change of length due to the application of the force) valid? What is the spring constant κ?

Help: The number of different permutations of m elements (order is important), taken from a set of n elements is $n!/m!$. The number of different combinations of m elements (order does not matter), taken from a set of n elements is $n!/m!(n-m)!$. For large n the approximation $\ln n! = n(\ln n) - n$ can be used.

SM&T II. (Mendez)

Consider N free electrons in a long metal wire of length L, which can be treated as a one-dimensional (1D), quantum-mechanical free electron gas.

a) (3 points) Calculate the number of states in the interval Δk, where k represents the wavenumber.

b) (7 points) Calculate the number of states in a small interval ΔE, where E refers to the energy of the electron. Show that the electronic density of states $g(E)$ can be written as

$$g(E) = \frac{L}{\pi\hbar}(2m)^{1/2}\frac{1}{\sqrt{E}}.$$

c) (4 points) Show that the Fermi energy of this 1D gas can be expressed as

$$E_F = \frac{\pi^2}{8}\frac{\hbar^2 n^2}{m},$$

where n is the number of free electrons per unit length.

d) (3 points) Estimate the Fermi energy for a copper wire, making an educated guess for n. Express your result in electron-volts.

e) (3 points) What should be the thickness of the copper wire in d) to consider it "one-dimensional" for the calculation of E_F? Have 1D wires been realized experimentally?

Fundamental constants: Electron charge, $e = 1.60 \times 10^{-19}$ C; Boltzmann constant, $k_B = 1.38 \times 10^{-23}$ J·K^{-1}; electron mass, $m = 9.11 \times 10^{-31}$ kg; Planck's constant, $h = 6.63 \times 10^{-34}$ J·s.

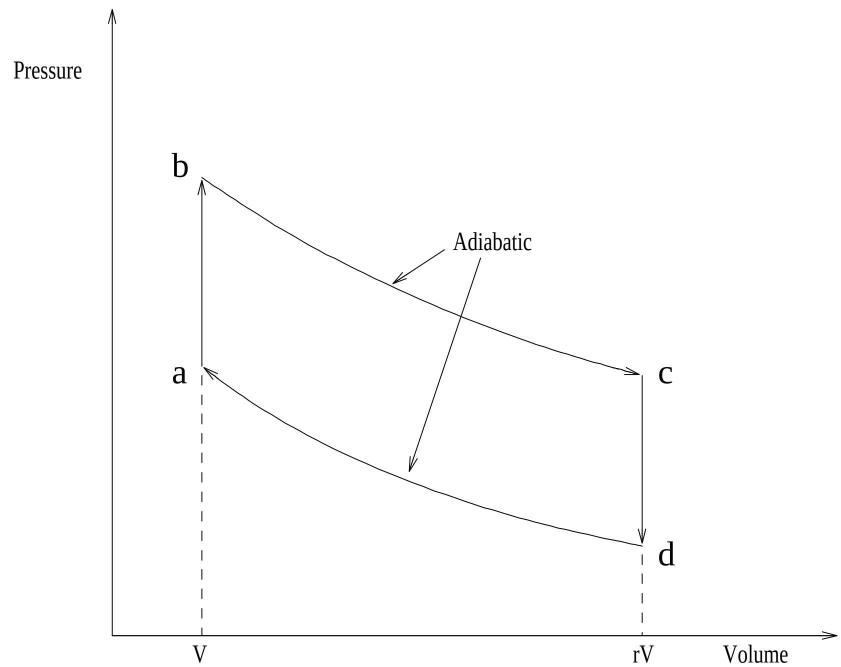


Figure 1:

SM&T III. (Hemmick)

Shown in the Figure is the so-called Otto cycle, an idealization of an internal combustion engine named after its inventor Nikolaus Otto who built and operated the first such engine in 1876.
The curve includes two adiabatic and two isochoric processes. The engine is operated between two defining volumes: V when the piston is positioned for minimum cylinder volume and rV for when the piston is positioned for maximum cylinder volume.

a. Derive the efficiency of this engine as a function of the volume ratio r. Recent car engines have pistons with shaped top surfaces that allow the piston to come very close to the valves thereby minimizing the volume V. Explain this design choice based upon your result.

b. Assume the number of degrees of freedom for the polyatomic gasses present during the operation of the engine is 7 (3 trans, 3 rot, 1 vib) and compute the efficiency using a compression ratio r=10. Compare this with the "industry best" of 30% efficiency.

c. Consider a Carnot Engine that operates between the same two extremes of temperature as your Otto engine (*i.e.* the T_H for the Carnot matches the temperature of the hottest point in the Otto cycle and the T_C for the Carnot engine matches the coldest point in the Otto cycle). Present a formula for the efficiency of the Carnot engine and demonstrate that its efficiency is higher than the result for the Otto cycle.