STONY BROOK UNIVERSITY
DEPARTMENT OF PHYSICS AND ASTRONOMY

Wednesday, 3 September 2003 – Day 2
Qualifying Examination in Quantum Mechanics
and in Statistical Mechanics and Thermodynamics

General instructions: Each area is graded separately. Do two of the three problems in one or both areas. Each problem is worth twenty points. If a problem has subparts, each of these will be equally weighted, unless indicated otherwise, with the sum totaling twenty points. Use one examination book per problem and label it carefully with your name, the name of the problem's author, and the date. You may not use any materials other than this examination paper and the exam books supplied, a calculator, your one page help sheet, and, with the proctor's approval, a foreign language dictionary. None of these materials may be shared between students.

Quantum Mechanics

Three problems, work any two.

QM I. (Hemmick) When a nucleus of charge Z is stripped of all surrounding electrons except one, it is called a hydrogen-like atom. If one neglects the nuclear radius, the normalized ground state wave function is given by

$$\Psi_{GS} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{\frac{3}{2}} e^{-\frac{Z}{a_0}r}. \tag{1}$$

Here a_0 is the Bohr Radius, $\frac{\hbar^2}{me^2}$. Assume that the nucleus is a uniformly charged spherical volume of mass number A, and radius $R_0 = 1.2 fm A^{\frac{1}{3}}$. NOTE: $R_0 << a_0$.

a. (4 points) Show that in the interior of a uniformly charged sphere of total charge Ze, the potential energy for a point charge of magnitude e varies with radius as

$$U_{in}(r) = -\frac{Ze^2}{2R_0}\left[3 - \left(\frac{r}{R_0}\right)^2\right]. \tag{2}$$

b. (10 points) Take the difference between the $\frac{1}{r}$ potential and the result from part **a.** as a perturbation. Calculate the change in energy of the ground state as a function of Z, e, R_0, and a_0.

c. (4 points) One can approximate that $A \propto Z$, for nuclei near the valley of stability. In this case, the energy shift from part (ii), will vary with nuclear charge as $\Delta E \propto Z^\alpha$. Determine the exponent α.

d. (2 points) If the electron were replaced by a muon, the ground state energy shift would be larger. Why?

QM II. (Verbaarschot)

We consider a hydrogen like atom with Hamiltonian given by $H = \mathbf{p}^2/2m + eA_0(\mathbf{x})$, where A_0 is the Coulomb potential and m the electron mass. The eigenstates of this Hamiltonian with eigenvalue E_n are denoted by $|n\rangle$. The absorbtion cross section for a classical radiation field with frequency ω polarized in the direction $\vec{\epsilon}$ is given by

$$\sigma_{\rm abs} = \frac{4\pi^2\alpha}{m^2\omega}|\langle n|\sum_k \epsilon_k p_k|i\rangle|^2\delta(\omega-\omega_{ni}), \tag{3}$$

where $\omega_{ni} = (E_n - E_i)/\hbar$ and α is the fine structure constant.

a) Show that the absorbtion cross section can rewritten as

$$\sigma_{\rm abs} = 4\pi^2\alpha\omega_{ni}|\langle n|\sum_k \epsilon_k x_k|i\rangle|^2\delta(\omega-\omega_{ni}). \tag{4}$$

b) Give selection rules for radiation polarized in the x direction.

c) What is the relation between the absorbtion cross section for the transition $|i\rangle = |l = 3, m = 3\rangle \to |n\rangle = |l = 4, m = 4\rangle$ and the transition from $|i\rangle = |l = 3, m = 2\rangle \to |n\rangle = |l = 4, m = 3\rangle$ and which of the two absorbtion cross sections is the largests? Ignore other quantum numbers.

QM III. (van Nieuwenhuizen)

Positronium is a bound system of a positron and an electron. Denote the mass of a free electron by m_e.

a. (5 points) What are the energies and degeneracies of the $n = 1$ and $n = 2$ Bohr levels of positronium? Use the notation $n^{2S+1}L_J$ to classify these bound states, where n is the principal quantum number, and S, L and J are the spin, orbital angular momentum, and total angular momentum of the combined system of the positron and electron. (The Bohr levels are the levels obtained from the Schrödinger equation if one neglects spin).

b. (5 points) What is the physical origin of the fine structure of the spectrum of the hydrogen atom? (Use no more than two sentences). Indicate in a figure which energy levels found in a. are split by this effect? (Do not calculate these splittings).

c. (5 points) What is the physical origin of the hyperfine structure of the spectrum of the hydrogen atom? (Use no more than two sentences). Indicate in the same figure as used in b. which levels found in a. which were not split by the fine structure are split by the hyperfine structure. (Do not calculate these splittings).

d. (5 points) Explain why the fine structure and the hyperfine structure are of comparable magnitude in positronium. List other effects which can lead to energy shifts.

Comment: a similar analysis is nowadays used to interpret the spectrum of charmonium (a $\bar{c}c$ bound system) and bottomium (a $\bar{b}b$ bound system).

Statistical Mechanics and Thermodynamics

Three problems, work any two.

SM&T I. (Shrock)

The Ising model in zero external magnetic field at a temperature T is defined by the partition function

$$Z = \sum_{\{S_i\}} e^{-\beta \mathcal{H}} \tag{5}$$

with the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle ij \rangle} S_i S_j \tag{6}$$

where $S_i = \pm 1$ are classical spins defined at each site of a lattice, J is the spin-spin exchange constant, and $\langle ij \rangle$ denote nearest-neighbor sites. Consider this model on a one-dimensional line of spins in the thermodynamic limit. You may choose any boundary conditions that you like. Let $\beta = (k_B T)^{-1}$ where k_B denotes the Boltzmann constant, and define $K = \beta J$. Restrict to the ferromagnetic case $J > 0$. By summing a high-temperature series expansion exactly or by using a transfer matrix (where the transfer matrix $\mathcal{T}$, has matrix elements $(\mathcal{T})_{ij} = \exp(K S_i S_j)$) or by other methods, calculate the following

- **a.** (5 points) the internal energy $U = \langle \mathcal{H} \rangle$.
- **b.** (5 points) the spin-spin correlation function $\langle S_0 S_\ell \rangle$.
- **c.** (5 points) the correlation length (first define this quantity carefully).
- **d.** (5 points) Also, by either of the methods mentioned above, or by any other method, calculate the magnetization and discuss its temperature dependence.

SM&T II. (Zahed)

A system of N noninteracting particles of mass m is enclosed in a container of volume $V = A\,x$. The bottom of the container is rigid and the top consists of an airtight piston of mass M that is allowed to slide without friction.

- **a.** (10 points) Derive the partition function Z for the system (gas-piston). You may neglect the effect of gravity on the gas molecules.
- **b.** (10 points) Show that the thermodynamical potential $-kT \ln Z$ is in the thermodynamical limit, identical to the Gibbs potential $G = \mu N$ of an ideal gas of N molecules with chemical potential μ, subject to a fixed pressure $P = Mg/A$.

Note: Stirling formulae $N! \approx N \ln N - N$.

SM&T III. (Abanov)

a. (6 points)Show that the entropy of an ideal gas of N atoms in a volume V at temperature T (up to a constant) is:

$$S \propto N \left[\ln \frac{V}{N} + \frac{3}{2} \ln T \right]$$

Where Boltzmann's constant k_B is taken to be one.

b. (14 points) What is the minimal work required to separate two different ideal monoatomic gases from the mixture of gases? Assume that the mixture is in thermal equilibrium at temperature T. The number of atoms of each gas in the mixture is N, the total volume does not change, and the pressure of the gases remains the same.