STONY BROOK UNIVERSITY

DEPARTMENT OF PHYSICS AND ASTRONOMY

Comprehensive Examination, January 24, 2007

General Instructions: Twelve problems are given. You should do any four, subject to the constraint that you must answer at least one from the experiment and one from the breadth category. Each problem counts 20 points, and the solution should typically take less than 45 minutes. Use one exam book for each problem, and label it carefully with the problem number and your name. You may use a one page help sheet, a calculator, and with the proctor's approval, a foreign language dictionary. No other materials may be used.

Here are a few useful numerical constants.
Avogadro number $N_A = 6 \times 10^{23}$ mol^{-1}
Boltzmann constant $k_B = 1.38 \times 10^{-23}$ J/K
Planck's constant $h = 6.62 \times 10^{-34}$ J s
Speed of light $c = 3 \times 10^8$ m/s
$\hbar c = 197$ MeV fm
Electric charge $e = 1.6 \times 10^{-19}$ Coul.
Gravitational constant $G = 6.67 \times 10^{-11} m^3/kg\, s^2$

Breadth I. Astronomy

This problem concerns the rotation curves (i.e., the relationship between velocity and radius) of galaxies. For a spherical distribution of mass of density, $\rho(r)$,

$$dM(r) = 4\pi r^2 \rho(r) dr, \tag{1}$$

where M is the mass within the distance, r, from the galactic center.

A. (5 points) The bulge region of a spiral galaxy contains a homogeneous distribution of stars. Find an expression for the velocity $V(r)$. To solve this problem, make the assumption that the stars are moving in circular orbits.

B. (5 points) Beyond the part of the rotation curve where the bulk of the luminous matter is seen, we might expect that the rotation curve becomes nearly Keplerian in nature. This, however is not what is observed. How are the rotation curves measured in the outer parts of spiral galaxies, and what is the observed relationship between $V(r)$ and r? What can be concluded from this result?

C. (5 points) Derive an expression for $\rho(r)$ in the outer parts of spiral galaxies. To do so, make the assumption that the dark mass has a spherical distribution.

D. (5 points) While spiral galaxies are rotationally supported, giant elliptical galaxies are not. Explain how the velocity structure of giant elliptical galaxies differs from that of spiral galaxies. Why are the disks of spiral galaxies believed to be formed via dissipational collapse, whereas elliptical galaxies are believed to be formed via dissipationless collapse?

Breadth II. Astronomy

Earth orbits the Sun at 1 AU (= 1.496×10^8 km). The Sun's "effective" temperature is $\mathrm{Teff}_\odot = 5800$ K, its radius is $R_\odot = 6.96 \times 10^5$ km and the Earth's radius is $R_\oplus = 6378$ km.

A. (7 points) Assuming the Earth absorbs all incident radiation and redistributes the energy uniformly over its surface before reradiating it, calculate its equilibrium temperature.

B. (7 points) Assume the Sun converts hydrogen to helium in its interior directly (ie., 100% from the PP I chain). What is the flux of neutrinos at the Earth? (Assume the mass deficit is 0.7%; the Stefan-Boltzmann constant is $\sigma = 5.67 \times 10^{-8}\,\mathrm{W\,m^2\,K^{-4}}$.)

C. (6 points) A main sequence F0 star ($\mathrm{Teff}_F = 7200$ K, $M = 1.6\,\mathrm{M}_\odot$, $R = 1.6\,\mathrm{R}_\odot$) also has an Earth in orbit. What would its orbital radius be if it had the sme temperature as our Earth? (Use the same assumptions as above). What would the neutrino flux at its surface be? (At this mass, assume energy is produced completely by the CNO cycle, where hydrogen is converted to helium through a series of reactions that involves C as a catalyst.)

Breadth III. Astronomy

Recent observations by the Hubble Space Telescope of Type Ia supernovae of redshift as large as $z = 1.755$ suggest that an energy component of negative pressure is present in the Universe at redshifts $z > 1$. Observations are so far consistent with a vacuum energy density of equation of state

$$p = -\epsilon \tag{1}$$

relating pressure p and energy density ϵ. More generally, the equation of state might be specified as

$$p = w\epsilon, \tag{2}$$

where present observations constrain

$$w \lesssim -0.7. \tag{3}$$

A. (10 points) Show that a "dark energy" equation of state $p = w\epsilon$ leads to a dark energy continuity equation

$$\epsilon = \epsilon_0 (1+z)^{-3(1+w)}, \tag{4}$$

where ϵ_0 is the present-day energy density of dark energy.

B. (10 points) Starting from the Friedmann equation for a spatially flat Universe

$$\dot{R}^2 = \frac{8\pi\rho G R^2}{3}, \tag{5}$$

where R is the scale factor, G is the gravitation constant, and ρ is the matter and energy density (in mass density units), show that the Hubble constant $H(z) \equiv \dot{R}/R$ is given by

$$H(z) = H_0 \left[\Omega_{m_0}(1+z)^3 + \Omega_{\Lambda_0}(1+z)^{-3(1+w)}\right], \tag{6}$$

where H_0, Ω_{m_0}, and Ω_{Λ_0} are the present-day Hubble constant, matter density parameter, and dark energy density paramter, respectively.

Breadth IV. Condensed Matter

Consider a semiconductor crystal with a periodicity defined by Bravais lattice vectors

$$\mathbf{a}_1 = a\hat{\mathbf{x}} \qquad \mathbf{a}_2 = b\hat{\mathbf{y}} \qquad \mathbf{a}_3 = c\hat{\mathbf{z}}$$

and a conduction band given by

$$\epsilon(\mathbf{k}) = \epsilon_0 + \epsilon_1 \cos(2k_x a) + \epsilon_2 \cos(2k_y b) + \epsilon_3 \cos(k_z c)$$

where ϵ_0, ϵ_1, ϵ_2, and ϵ_3 are all positive constants. At room temperature, a small number of electrons will occupy the lowest energy states in the conduction band. At time $t = 0$, a uniform DC electric field of magnitude E is applied in the $-\hat{z}$ direction. Consider the semiclassical dynamics of an electron that starts at the center of one of these electron pockets. Neglect scattering and interband transitions.

A. (7 points) Calculate the electron's velocity as a function of time.

B. (7 points) How long will it take the electron to have its first stop? How far, and in what direction, will it have travelled from $t = 0$ to the time of its first stop?

C. (6 points) How would you characterize the current response to this DC electric field? Why will it be difficult to observe this effect in all but the cleanest of solid-state samples? In what atomic physics experiment has an analogous effect recently been observed?

Breadth V. Elementary Particle

A. (5 points) The pion is the lightest strongly-interacting particle. The nucleon is the basic constituent of atomic nuclei. According to the quark model, each of these particles must contain certain quarks (including antiquarks). List the quarks in the three families or generations, giving their quantum numbers conserved by strong interactions, including but not limited to electric charge, spin and isospin.

A. (5 points) Which quarks are necessarily found in the three different pions? State the same for the two nucleons. Describe the simplest allowed orbital angular momenta of the quarks in these states, and explain how the quantum statistics constraints on the quarks in the nucleon states are satisfied.

B. (5 points) The charged and neutral pions have somewhat different masses. What is believed to account for this difference? Electroweak interactions may have to be taken into account. In a different example of such splittings, the neutron is heavier than the proton. How has this been explained? Does this mechanism have any influence on the pion mass splitting? Explain your answer.

C. (5 points) Could a quark other than the ones listed in item (b) be found in a pion or a nucleon? Explain your reasoning, including how such a constituent would be consistent with the quantum numbers of the pion or nucleon.

Breadth VI. Nuclear Physics

As a first approximation the proton-neutron force can be described by an attractive square well potential with $V(r) = -V_0$ for $r \leq a$ and zero otherwise. The range of the corresponding force is of the order of the Compton wavelength $\lambda_\pi = \hbar/(m_\pi c)$ with the pion mass $m_\pi = 139$ MeV and $\hbar c = 197$ MeV-fm.

A. (5 points) What is the minimum depth of the potential for which there is one bound state?

B. (5 points) What bound could be put on this depth if there were no Deuteron (proton-neutron) bound state? Use $m_n = m_p = 940$ MeV.

C. (5 points) Derive a precise relation between the binding energy E of the Deuteron and the depth of the potential V_0.

D. (5 points) Improve on your estimate in (b) by using the fact that $E = -2.22$ MeV.

Note: The Laplacian $-\hbar^2\nabla^2$ in spherical coordinates is $-\frac{1}{r}\partial_r^2\, r + \frac{L^2}{r^2}$ where L is the orbital angular momentum.

Experiment I. Astronomy

(20 points) Distances are among the most fundamental quantities that astronomers need to know. The point of this exercise is to determine the length of the astronomical unit (distance from Earth to Sun), as astronomers first did in 1761 and 1769. The proctor will give you a ruler if you want.

The pictures on the following pages show the transit of Venus in 2004 as observed from observatories in Denmark and Australia. There are two images of Venus silhouetted against the Sun. A line drawn between the two is the orbital path of the planet. Due to parallax, the two orbital paths are shifted somewhat N-S. Measure this angular shift, and determine the Astronomical Unit, (The E-W shifts are a consequence of different times of observation, and are unimportant.)

All you need to know is:

- The latitude of the observatory in Denmark: 55°45'
- The latitude of the observatory in Australia: -22°15'
- The radius of the Earth: 6378 km
- The angular diameter of the Sun: 31' 59.3"
- The orbital period of Venus: 224.7 days
- Kepler's third law: $P^2 = a^3$

Provide an estimate of the length of the Astronomical Unit in kilometers, along with an uncertainty. Include a full discussion of the sources of uncertainty.

Experiment II. Elementary Particle

A stir was created when it was reported that an experiment had discovered the decay of $\mu^+ \to e^+ + \gamma$ at a branching ratio $\approx 10^{-9}$. It turned out not to be true.

A. (5 points) What general principle is believed to be responsible for the suppression of this decay?

B. (10 points) The apparatus to measure such a decay consisted of equipment to stop the μ^+ beam and two NaI crystals, which respond to the total energy of the positrons and gamma rays. Explain how you would arrange the crystals relative to the stopping target and the beam, and what signal in the crystal would indicate that an event is such a *mu* decay?

C. (5 points) The background events are the decays $\mu^+ \to e^+ + \nu_e + \overline{\nu}_\mu + \gamma$. Describe qualitatively how one would distinguish events of this type from the signal you were looking for in this experiment.

Experiment III. Nuclear

The form factor for the ^{16}O nucleus is

$$F(q^2) = \exp(\frac{-q^2}{4\alpha^2})(8 - \frac{q^2}{\alpha^2})$$

A. (6 points) Explain how the form factor is measured.

A. (7 points) Generally, what nuclear properties do the first two terms

$$Z - \frac{q^2\langle Zr^2\rangle}{6}$$

in the expansion of the form factor correspond to?

A. (7 points) What is the r.m.s. charge radius of ^{16}O in terms of α?

Experiment IV. Atomic, Molecular, and Optical

Many experimental setups in ultracold-atom laboratories feature a so-called Zeeman slower. A Zeeman slower consists of a vacuum tube (connected to an oven) in which fast atoms are decelerated with the radiation pressure of a counterpropagating laser beam, which is kept on resonance with an atomic transition. The Zeeman slower produces a slow atomic beam that can *e.g.*, be used to efficiently load an atom trap.

Let's consider a beam of ^{87}Rb atoms ($m = 1.4 \times 10^{-25}$kg, transition wavelength $\lambda = 780$nm (D_2 line) and natural line width $\Gamma = 2\pi \times 6.1$MHz) escaping from an oven at temperature T=110°C.

Note that each part contains several independent questions.

A (5 points). (1) What is the *average* transfer of momentum that occurs when an atom in the beam scatters a photon (i.e. absorption followed by spontaneous re-emission)? (2) What is the change in its velocity as a result of this momentum transfer?

B (5 points). (1) Give an expression for a typical thermal velocity v_{th} of atoms in the oven. (2) How many photon scattering events are needed to stop atoms escaping with this velocity? (3) What is the maximum deceleration that one can achieve, and what is the minimum distance L needed for stopping these atoms? (assuming arbitrarily high intensities in the laser beam)

C (5 points). Suppose the laser beam is resonant with atoms entering the slower at a design velocity $v_c := v_{th}$. Because of the Doppler frequency shift $\delta\omega_D = \vec{k} \cdot \vec{v}$ (where $k = 2\pi/\lambda$) the atoms will be shifted more and more out of resonance as they slow down. This unwanted effect can be eliminated by applying a longitudinal magnetic field $B(x)$ that Zeeman shifts the transition frequency by $\delta\omega_Z = \tilde{g}\mu_B B(x)$ back onto resonance ($\tilde{g}\mu_B = 2\pi \times 1.4$ MHz/Gauss for the transition considered). Calculate the spatial dependence of $B(x)$ for the optimum slower.

D (5 points). An important feature of the Zeeman slowing technique is that it not only decelerates but also *cools* the beam, i.e. it compresses the atomic velocity distribution along the beam direction. Explain why this is the case. (Hint: consider atoms that enter the slower with velocities smaller and larger than v_c).

Experiment V. Condensed Matter

This problem has to do with finding the optimal conditions to measure a small electrical signal, and the ultimate achievable sensitivity.

A (4 points). Due to thermal fluctuations, a noise voltage appears across any resistor, called Johnson noise. What is the root-mean-square (rms) value of V_{noise} measured in a bandwidth Δf for a resistor R at temperature T?

B (4 points). The performance of low-noise amplifiers is specified in terms of the noise figure, which is the ratio of the output voltage when the amplifier is connected to a specified resistor at room temperature to the output voltage produced by noise from the amplifier alone. Noise figure is generally measured in decibels ($dB = 20\log_{10}$(voltage ratio)). If the noise figure of a particular amplifier is 0.05 dB for a source with impedance 100 kΩ, what is the equivalent noise voltage in a 1 Hz bandwidth contributed by the amplifier?

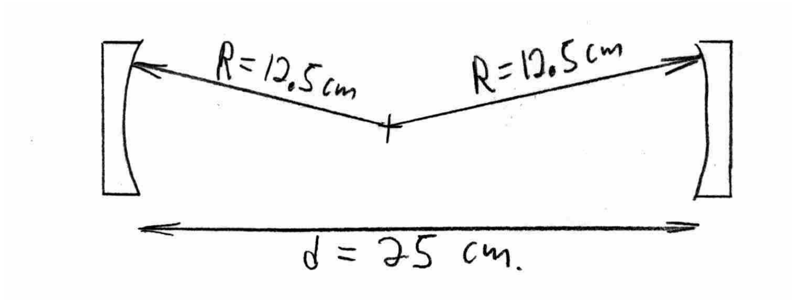
C (4 points). A carbon resistor is a convenient thermometer at cryogenic temperatures. The resistance of a particular resistor for temperature T in the neighborhood of 4 K is $R(T) = R_0 \exp(A/T)$ where $R_0 = 16\text{k}\Omega$ and $T = 4\text{K}$. This is used as one element of a Wheatstone bridge, where the other three resistors are independent of temperature. The bridge is driven by a 1kHz sine wave of adjustable amplitude V_d. You may assume the off-balance signal of the bridge is given by $\delta V = (V_d/4)\,\delta R/R$. You may also assume that all of the resistors are at a low temperature, and ignore their Johnson noise. The amplifier noise (same amplifier from part B) looks like a noise in the measured temperature, ΔT_{rms}. What is the uncertainty ΔT_{rms} in a bandwidth $\Delta f = 1\text{Hz}$ due to the amplifier for $T = 4K$, as a function of V_d?

D (4 points). The bridge drive voltage is limited by the condition that the resistor should not be heated significantly above the object to be measured. Take the effective thermal link to be given by $P = \kappa\,\Delta T$ with $\kappa = 0.01$ W/K. Give an expression for $\Delta T_{heating}$ as a function of V_d.

E (4 points). What is the optimum drive voltage to minimize the error in temperature measurement, balancing amplifier noise with self heating? What is the smallest temperature change that can be measured with an accuracy of 10% under these conditions, if the bandwidth of the measurement is to be 1 Hz?

Experiment VI. Atomic, Molecular, and Optics

Consider the laser cavity shown below.



A. (5 points) Copy the diagram into your answer book and draw phase fronts (indicate the position of the waist and radii of curvature at the mirrors and waist) for a Gaussian beam transverse mode of the cavity.

B. (5 points) What is the frequency spacing between longitudinal modes of the cavity?

C. (10 points) Determine whether there exist stable transverse eigenmodes of the cavity. The relevant ABCD matrices for Gaussian beam and ray propagation are:

Mirror with radius of curvature R : $\begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix}$

Free space propagation through distance d : $\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$