a. In atomic units, the energies are $-1/2n^2$. Lyman $\alpha$ corresponds to $-1/8 + 1/2 = -3/8$. The atomic unit of energy is a Hartree (27.2 eV) and so $L_\alpha$ energy is 10.2 eV. This works out to about 121.5 nm.

b. The spin orbit energy is given by $-\mu_e B$. The magnetic field felt by the electron (arising from the motion of the electron in the electric field of the proton - $E = \frac{Ze^4}{4\pi\varepsilon_0 r^2} \hat{r}$) is given by $B = \frac{\mu_e}{c r} \times E$. The magnetic moment of the electron is $\mu_e = g\mu_B s$, where $\mu_B = \frac{e\hbar}{2m}$. This gives rise to an (uncorrected) spin orbit energy of $U = (\frac{e\hbar}{mc})^2 \frac{Z^2}{4\pi\varepsilon_0 r^3} L \cdot s$, where $L = r \times p$. The 'Thomas' correction, which accounts for the non-inertial character of the electron’s frame yields: $U = \frac{1}{2} (\frac{e\hbar}{mc})^2 \frac{Z^2}{4\pi\varepsilon_0 r^3} L \cdot s$. Plugging in the constants (noting that for $l=1$ the $L \cdot s$ term is order unity), we get $U \sim 1.7 \times 10^{-23}$ J or about $h \times 1$ GHz.

c. Hyperfine splitting is about 3 orders of magnitude smaller than spin-orbit splitting because the magnetic moments of nuclei are about 3 orders of magnitude smaller than the magnetic moment of electrons.
**Fall 2007 comprehensive exam problem, condensed matter, breadth.**

**(a,b)** Graphene is a single (infinite, 2d) sheet of carbon atoms in the graphitic honeycomb lattice. On the left is a fragment of the lattice showing a primitive unit cell,

![Graphene lattice fragment](image1)

with lattice parameter \(a\), primitive translation vectors \(a\) and \(b\), and corresponding primitive vectors \(G_1\), \(G_2\) of the reciprocal lattice. These vectors can be written as

\[
\begin{align*}
\bar{a} &= a\left(\frac{3}{2},-\sqrt{3}/2\right) \\
\bar{b} &= a\left(\frac{3}{2},\sqrt{3}/2\right) \\
\bar{G}_1 &= (2\pi/3a)(1, -\sqrt{3}) \\
\bar{G}_2 &= (2\pi/3a)(1, +\sqrt{3})
\end{align*}
\]

Notice that there are two atoms in the primitive cell, one occupying corners and one in the interior of the unit cell shown in red. On the right is the central part of the reciprocal lattice and the first Brillouin zone.

**(c)** The corners of the Brillouin zone are the points \(K_i\) given by

\[
\begin{align*}
\tilde{K}_1 &= (\tilde{G}_2 - \tilde{G}_1)/3, \\
\tilde{K}_2 &= (2\tilde{G}_1 + \tilde{G}_2)/3, \\
\tilde{K}_3 &= (2\tilde{G}_1 + \tilde{G}_2)/3, \\
\tilde{K}_4 &= (\tilde{G}_2 - \tilde{G}_1)/3, \\
\tilde{K}_5 &= (\tilde{G}_1 - \tilde{G}_2)/3, \\
\tilde{K}_6 &= (\tilde{G}_2 - \tilde{G}_1 - \tilde{G}_2)/3.
\end{align*}
\]

Notice that every corner is either \(\tilde{K}_1\) or \(\tilde{K}_2\).

**(d)** The electron density of states, per unit cell is defined as

\[
D_\pm(e) = \frac{2}{N} \sum_\mathbf{k} \delta(e \mp v|\mathbf{k}|).
\]

The factor of 2 is for the two different zone corners. This is for one spin orientation. Convert the sum to an integral:

\[
D_\pm(e) = \frac{2A}{(2\pi)^2} \int_{0}^{2\pi} d\phi \int_{0}^{\pi} kdk \delta(e \mp vk) = \frac{3\sqrt{3}}{\pi} \left(\frac{a}{\hbar v}\right)^2 |e|,
\]

where \(A\) is the area of a unit cell, \(3\sqrt{3}a^2/2\).

**(e)** The temperature interval 0.1 to 10K is one where phonon contributions to specific heat scale as \(T^3\) and are small, while electrons can be treated as a degenerate Fermi gas since \(E_F\) is large compared to \(kT\). The specific heat is

\[
C = \left(\frac{\pi^2}{3}\right) k_B^2 D(e_F) k_B T
\]

for degenerate electrons, so \(C(T)\) is linear in \(T\) with the slope which scales as the magnitude of the Fermi energy, independent of its sign.
Nu clear physics breadth

(a) \( I = 0 \Rightarrow \text{antisymmetric isospin w.f.,} \ (-1) \) when proton
\( T = \frac{1}{2} \), \( J = 1 \) \( \Rightarrow \text{spin, isospin, orbital} \Rightarrow 6 - 1 \)
\( (-1) \)

Thus \( \text{(spin, orbital)} = 4 + 1 \)

If spin 1, (\( \Sigma \), symmetric)
then orbital \( r^{-1} = 1 \), \( L = 0, 2 \), different from p
positive \( P = + \)

But, since \( J = 1 \), only 0, 2 allowed

\( I = 1 \) means isospin w.f. gives \( +1 \) when p\( \rightarrow n \):
So, if spin \( S = 1 \) then \( \bigwedge = 1 \)
if \( S = 0 \) then \( \bigwedge = 0 \), 2

\( \Phi \): outside \( \frac{-\alpha r}{e} \)
Schr. eqn
\( \left(-\frac{\hbar^2}{2m*} \nabla^2 + V \right) \Phi = E \Phi \)

\( x = \frac{-\hbar^2}{2m*} \alpha^2 + |B| \alpha e = 0 \)

\( x^2 = \frac{mB}{\hbar^2} \approx 490 \text{ MeV} \cdot 2.2 \text{ MeV} \)

\( x \approx 4.5 \text{ MeV} \cdot \alpha \)

inside \( \int e^{-\alpha r} r^2 dr = \frac{\alpha^3}{3} \)

if we ignore the potential

outside \( \int e^{-\alpha r} r^2 dr = 2x^3 \)

Part outside \( \approx \) outside

\( \frac{\text{total}}{1 + e^{-\alpha a^3 \alpha^3}} = \frac{1}{1 + \frac{1}{6} e^{-0.23}(0.23)^3} \)
If \( x = 0 \) (zero binding) then

\[
K_0 = \frac{\sqrt{t}}{2a}, \quad K = \frac{\sqrt{t}}{2a} \approx 1.57
\]

\[
V = \frac{k^2}{m} = \frac{(\frac{m}{2a})^2}{m} = \frac{m}{2} \left( \frac{\sqrt{t}}{2} \right)^2 \left( \frac{197 \text{ fm} \cdot \text{MeV}}{4 \text{ fm}^2} \right)^2 = 101 \text{ MeV} \approx 308 \text{ MeV}
\]

0 and 5 2.2 MeV

\[
\frac{309 \text{ keV}}{2.2 \text{ keV}} = 140 \quad \text{while} \quad \sqrt{\frac{13}{2}} = 1.57
\]

So difference in \( V = K \) is \( \approx 2\% \) only

Possibilities:

- \( n + p \)
- \( n + p \rightarrow \text{very easy, no Coulomb} \)

- \( t + p \rightarrow \text{He}^4 \)
- \( t + n + \text{He}^5 \) (doubt exist)
- \( \text{He}^4 + \text{He}^5 \rightarrow \text{He}^6 \)

Stellar: Stars like Sun would do \( \text{H, He... Bnurnings} \)

\( n \) more rapid
High Energy Physics Breadth Solution:

The hydrogen formula is \( E = -\alpha m_{\text{electron}} c^2 / 2n^2 \), where in this case for the successive states we should have \( n = 1, 2, 3 \). Of course, this is not hydrogen, but if we assume the short-range force is Coulombic (though of course with a bigger \( \alpha \) than for hydrogen), and use a reduced mass \( m = m_c / 2 \), we get a ratio of the second interval to the first of \((1/4 - 1/9)/(3/4) = 5/27 = 0.185\). The corresponding ratio for the masses above is 0.144, just a bit smaller. Here are two factors which can help explain this: First, in addition to a Coulombic potential there is a piece rising approximately linearly with radius, that gives closer to equal spacing. That in itself would go the wrong way, but the linear potential is cut off at the threshold for \( D\bar{D} \) production, and this goes in the right direction. Also, the relativistic kinetic energy formula becomes more important for the higher states, and reduces their energy.

Astronomy Breadth 1 Solution:

1. The pressure is
   \[
   P = nkT. \tag{1}
   \]
   The pressure exerted by the weight \( W \) of a column of length \( L \) is
   \[
   W/L^2 = G(nmL^3)^2/L^2 = G n^2 m^2 L^2. \tag{2}
   \]
   Equating equations (3) and (4) evaluated at \( L = L_J \) gives
   \[
   nkT = G n^2 m^2 L_J^2, \tag{3}
   \]
   which is solved for \( L_J \) as
   \[
   L_J = \left( \frac{kT}{Gn} \right)^{1/2} m^{-1}. \tag{4}
   \]
   or for \( M_J = nmL_J^3 \) as
   \[
   M_J = \left( \frac{kT}{G} \right)^{3/2} n^{-1/2} m^{-2}. \tag{5}
   \]

2. The radiation temperature is given by
   \[
   T = T_0 (1 + z), \tag{6}
   \]
   and the matter density is given by
   \[
   n = n_0 (1 + z)^3, \tag{7}
   \]
   so at recombination
   \[
   M_J = \left( \frac{kT_0 (1 + z)^3}{G} \right)^{3/2} \left[ n_0 (1 + z)^3 \right]^{-1/2} m^{-2} \tag{8}
   \]
   \[
   = \left( \frac{kT_0}{G} \right)^{3/2} n_0^{-1/2} m^{-2}. \tag{9}
   \]
3. Using the specified values, the present-day proton density is

\[ n_0 = \eta n_0(\text{photon}) = 2.6 \times 10^{-7} \, \text{cm}^{-3}. \]  

(10)

Using the other specified values, the Jean’s mass at the epoch of recombination is

\[ M_J = 2.9 \times 10^{38} \, \text{g} = 1.4 \times 10^5 M_\odot, \]  

(11)

which is comparable to the mass of a globular star cluster.

**Astronomy Breadth 2 Solution:**

a. Do the math. 0.007 of the rest mass is converted to energy in each reaction. Use \( E=mc^2 \) to find that each reaction generates \( B=1.4 \times 10^{-5} \) ergs. The number of reactions needed to generate the solar luminosity is \( L_\odot/B = 2.8 \times 10^{38} /\text{sec}. \) It takes \( 4.27 \times 10^{17} \) seconds (1.4\times10^{10} \) years) to convert the solar core to helium. A 20 solar mass B star will be about \( 20^3 \) times as luminous as the Sun. The lifetime is proportional to \( M/L, \) so is \( 1.4 \times 10^{10} \) years \times 20/20^3, or 35 million years.

b. The virial theorem states that, for a system in equilibrium, \( 2K+U=0 \) where \( K \) is the kinetic energy and \( U \) is the potential energy in the system.

The kinetic energy is \( \Sigma 1/2mv^2. \) \( v \) is the three dimensional velocity dispersion, which is \( \sqrt{3} \) times the one dimensional velocity dispersion. The total kinetic energy is therefore \( \sqrt{3} M, \) where \( M \) is the total mass of the system [units of mass (km/s)^2].

The gravitational potential energy is \( 1/2 GM/R, \) where \( M \) is the total mass and \( R \) is a typical radius of the association.

Plug in the numbers and equate the two sides. The system is not gravitationally bound.

c. Galactic tides will disrupt the association if the tidal force across the association \( (GM_{gal}/[1/D^2-(1/(D\pm R)^2)] \) exceeds the gravitation self-attraction of the cluster \( (GM/R^2). \) Plug in the numbers. Tides win.

Ambartsumian was right. Associations are ephemeral because the O and B stars that make them noticeable run out of fuel and blow up on timescales of a few tens of millions of years, and meanwhile the associations are expanding into the Galaxy while also being tidally disrupted. A velocity dispersion of 2 km/s increases the association size by about 2 pc in a million years. A 10 Myr old association has expanded from close to a point source in 10 Myr.
Atomic, Molecular, and Optical, Experiment

1. \[ E = \frac{3}{4}Ry(Z - \sigma)^2 \] \(Ry\) is the Rydberg constant (13.6 eV), \(\sigma\) is a parameter, approximately unity, which accounts for the presence of a second 1s electron (sometimes considered a screening constant) and \(Z\) is the atomic number. The scaling with atomic number is quadratic because while the potential for \(Z\) charges scales linearly with \(Z\), the radius of a given orbital is inversely proportional to \(Z\). This results in a quadratic \(Z\) dependence for the energy difference.

2. 1s2s2p63s23p64s13d10. For a \(K_{\beta}\) x-ray, an electron drops from the M shell into the K shell. For an \(L_{\alpha}\) x-ray, an electron drops from the M shell into the L shell. Fine structure (spin orbit coupling) leads to the splitting of the \(K_{\alpha}\) line into 1 and 2.

3. First a K shell electron has to be kicked out, meaning that it has to be promoted to an unfilled orbital or to the continuum. Then an L shell electron can drop down to the K shell with the atom emitting an x-ray whose energy corresponds to the energy difference between the K and L shell orbitals.

4. Moseley’s law works almost exactly for Cu with \(\sigma = 1\); solving

\[ 8050 = 13.6 \times 3/4 \times (Z - \sigma)^2 \]

gives \(Z - \sigma = 28.1\). Emboldened, we plug in \(Z = 41\) and get 16.32 keV. (The measured value is 17.48 keV, which implies \(\sigma = 0.6\).)

5. Inner shell electrons don’t participate in molecular bonds and their energies are only slightly perturbed by the surrounding atoms. Air is transparent to \(K_{\alpha}\) X-rays from elements with \(z=21-50\), and the transition energies are well above \(K_T\) at room temperature.

Condensed Matter Experiment

1. Somewhat counterintuitively, the measured sheet resistance is \(1/\sigma d\), independent of \(a\). Current flows a distance that scales as \(a\) through a cross section that scales as \(ad\), so \(R \propto a/(ad\sigma)\).

2. Hall effect, \(V_{\text{Hall}} \propto H/d\) independent of \(a\).

3. By dimensional analysis, \(Q\) must be inverse conductivity, i.e., length times resistance. The impedance of free space is \((\mu/\varepsilon)^{1/2} = 377\Omega\), so \(Q = d(\mu/\varepsilon)^{1/2}\). The transmission has this perhaps unexpected form because the radiation not transmitted is reflected, not absorbed.

4. \(R = 1/d\sigma \sim 377\Omega\), so \(d \sim (377\Omega \times 10^6\Omega^{-1}m^{-1})^{-1} \sim 30\) nm. The connection to part 1 is now clear, because one will get a transmitting sample if the measured sheet resistance is on the order of 377 \(\Omega\).
a) \[
\bar{\Delta \phi} = \frac{(d\phi}{dp}) \bar{\Delta p} \Rightarrow \bar{\Delta \phi} = \frac{0.3 S_{\text{Be}}}{p^2} \bar{\Delta p}
\]

or \[
\frac{\bar{\Delta p}}{p} = \frac{\bar{\Delta \phi}}{\Delta \phi} = \frac{\bar{\Delta \phi}}{0.3 S_{\text{Be}}}
\]

\[
\bar{\Delta \phi} = \frac{12}{1} \text{ mrad}
\]

\[
S_{\text{Be}} = 2 \text{ cm}
\]

\[
\Rightarrow \frac{\bar{\Delta p}}{p} = \frac{1}{12} p \text{ (in GeV)}
\]

b) MWPC "multi wire proportional chambers"

- Ionize gas, amplify around thin wire, register wire hit
- Combine several layers for tracking
- Resolution $\frac{1}{12}$ channel separation

DC drift chambers

- Ionize gas, drift at constant drift velocity to anode wire, amplify
- Measure time and current back to position
- Resolution $\approx 100-300 \mu$m
9) measure $t_{oto}$ with $t_{tof}$ = 80 ps

expected time

expected time $t_{eto} = \frac{L}{\beta c} = \frac{\sqrt{p^2 + m^2}}{p} \frac{L}{c}$

find momentum at which $t_{eto} = t_{eto}$

$3 \times 80 = 240$ ps

$\sqrt{p^2 + m^2} = \frac{240ps \cdot c}{L}$

assume $\beta = 1$

$K = \frac{\sqrt{p^2 + m^2}}{p} \frac{1}{\sqrt{1 - \frac{m^2}{p^2}}}$

$K^2 = p^2 + m^2 \Rightarrow \frac{m^2}{p^2} = \frac{1}{K(2 - K)K}$

$p \approx 5.5$ GeV

$p = \frac{m_p}{\sqrt{2K}}$
1) Cherenkov detectors:

- Use Cherenkov effect:
  \[ \cos \theta = \frac{v}{c} \]
- Threshold for particles to fire Cherenkov light:
- High momentum good for electron id

- \( \frac{dE}{dx} \)

\[ \text{Belle Block} \]

- Relativistic rise

- Measure ionization

- Distinguish particles by different amount of energy loss (check many samples)
High Energy Experiment

Solution:

a. (a) One only needs to detect scattered muon and its momentum: Tracking chambers + Spectrometer Magnet + Muon Detectors

(b) One needs to detect scattered muon and target fragments: [detector components in (a)] + particle ID detectors.

(c) A wide acceptance detector to detect scattered muons, target fragments and beam fragments: Detector as in (b) with wide geometric acceptance to at least cover angles up to ±90° with respect to the initial beam direction. The main difficulty in the measurement beyond the detector acceptance is proper identification of beam and target fragments. This is essentially impossible to do in a fixed target experiment, but can be done fairly easily in a collider experiment.

b. The square of the four-momentum transfer \(q^2 = (k_i^2 - k_f^2)\), where \(k_i/f\) are the initial/final four momenta of the target quark, respectively. In the rest frame of the quark, they are given by:

\[k_i = (m_q, 0, 0, 0)\text{ and } k_f = (m_q + \nu, p, 0, 0)\]

where \(m_q\) is the quark mass, and \(p\) is the momentum in x direction it acquires after collision. So:

\[q^2 = \nu^2 - p^2 = \nu^2 + m_q^2 - E_q^2\]

where \(E_q = m_q + \nu\) is the final energy of the quark and we have used the relation \(E_q^2 = p^2 + m_q^2\). Substituting \(E_1\) we get,

\[q^2 = -2m_q\nu = -2x m_N\nu\]

, which we can re-arrange to find \(x\) in terms of the nucleon mass \(m_N\):

\[x = \frac{-q^2}{2m_N\nu}\]

c. We expect incoherent scattering from quarks, so that we may sum the contributions to the cross section from each type of quark present in the target. Since the scattering will be dominated by photon exchange, the scattering from a given quark flavor is proportional to the squared charge of that quark. Therefore a proton, which comprises two \(u\) quarks and one \(d\) quark, has a cross section for scattering a muon of:

\[\sigma_{\mu-p} \propto \sum Q_i^2 = 2 \times Q_u^2 + Q_d^2 = 2 \left(\frac{2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 = 1\]

where \(Q_i\) is the charge of a quark of type. On the other hand, the deuteron consists of 3 \(u\) quarks and 3 \(d\) quarks, so that for scattering from deuterons,

\[\sigma_{\mu-d} \propto 3Q_u^2 + 3Q_d^2 = \frac{15}{9} = \frac{5}{3}\]
So we expect the ratio of cross-sections to be

$$\frac{\sigma_{\mu-p}}{\sigma_{\mu-d}} \sim \frac{9}{15}$$

d. the rest of the momentum is carried by gluons, which only interact through strong force. Muons are leptons, which do not interact by the strong force. (Virtual photons are EM charge carries without color, and gluons are charge neutral, so neither have in inclination to interact with each other, as well).

Astronomy Experiment 1 Solution:

a. What would be the baseline required to make such a measurement? Be careful to define what quantity an interferometer measures and what feature of that measurement you’d use to determine the diameter (1 pc ~ 3.1 × 10^{16} m). (5 points)

Answer: The quantity measured is the “Visibility” or fractional modulation of the interference fringes,

$$V = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}}.$$ 

The usual measurement is to vary the baseline of the pair of apertures, shorter to longer, until the visibility - modulation - first goes to zero. This occurs at about $$B\theta/\lambda = 1$$, allowing the determination of the angular diameter, which is $$\theta = 5.4 \times 10^{-15} \text{ rad} = 1.1 \times 10^{-9} \text{ nano arc-seconds}$$. From above this would require a 1000 km baseline at 50 Å.

b. Assuming a surface temperature of 30 ev, collecting areas of 1 m^2 for each aperture, and a bandpass of 1 Å, how much time would it take to measure a diameter accurate to 10%? Assume that a diameter accurate to 10% is equivalent to making three measurements at suitable baselines, each accumulating 1000 photons, that the interferometer/detector system is 100% efficient and that there is no intervening absorption ($$B_\lambda d\lambda = \frac{2\pi \hbar c}{\lambda^2} (\exp\{\hbar c/\lambda kT\} - 1)^{-1}$$). (5 points)

Answer: Assuming the star is spherically symmetric, the energy flux received (per unit area) is just

$$f_\lambda \Delta\lambda = B_\lambda \Delta\lambda \left(\frac{R}{d}\right)^2.$$ 

At a temperature of 30 ev, a wavelength of 50 Å, and with an angular diameter of 5.4 × 10^{-15} rad, this evaluates to 2.34 × 10^{-17} W m^{-2}. For two telescopes of about a square meter each, this is equivalent to a mean photon rate of 1 Hz. 3000 photons takes 50 minutes.

c. RXJ185635-3754 is actually thought to have a hot spot that produces most of the flux at 50 Å. Assuming a uniform spot of 2 km radius with a temperature of 60 ev, what fraction of the flux received is from the spot? If you wanted to measure the spot’s size directly, how long would the baseline have to be, how long would it take? (5 points)
Answer: The spot is a two-tenths the size of the star, and hence would require five times the baseline. The flux from the spot would be 2.4 times the flux from the rest of the star or about 70% of the total. In turn the measurement would be 2.4 times faster, assuming the extra baseline was available.

d. The cooler surface of the star rapidly dominates the flux as one goes to longer wavelengths. Assuming that, what would be the best wavelength to measure the diameter of the star itself, at least in terms of the emitted flux? Why do you think they aren’t planning to make the measurement in that wavelength region? (5 points)

Answer: In wavelength units the peak of the Planck function is at \( \frac{hc}{\lambda kT} \sim 5 \) (more accurately, \( \lambda T = 2500 \, \text{Å} \cdot \text{ev} \)). For a temperature of 30 ev the Planck maximum is thus at about 83 Å. However, interstellar absorption, due to neutral hydrogen mostly, quickly becomes optically thick in this wavelength region, even at these small distances. The star would in fact appear brighter at 50 Å!

**Astronomy Experiment 2 Solution:**

a. The diffraction limit is \( \lambda/D \) (radians). For \( \lambda \) in \( \mu \text{m} \) and \( D \) in m, conversion of radians to arc sec gives the numerical answer \( 0.2 \times \lambda_{\mu\text{m}}/D \)

b. \( \text{pixelsize} / \text{focal length} = \text{field of view of pixel} \)
   \[ \text{pixelsize} / fD = 0.5\lambda/D \]
   \[ f = \text{pixelsize}/(0.5\lambda) = 18.5/1 = 18.5 \text{ meters.} \] This does not depend on \( D \).

c. How many arcseconds across (on sky) is each pixel, and what is the field of view across a side of a detector? 0.03 arcsec and 60 (or 63) arcsec.

d. The total light power reaching the detector goes as \( D^2 \), and the peak intensity goes as \( D^4 \) (because linear size of image \( \propto 1/D \)).

e. area = 33.19 m\(^2\), \( F = 1.5 \times 10^9 \), \( Q = 0.5 \), \( t = 2 \) sec, fullwell = \( 1.0 \times 10^5 \), frac = 0.13 in one pixel.
   \( \frac{\text{counts for zero mag star in 2 seconds}}{\text{fullwell}} = \text{area} \times F \times \text{qe} \times t \times \text{frac} / \text{fullwell} = 6 \times 10^4 \) which is \( 2.5 \log_{10}(60000) \) astronomical magnitudes, about 12th magnitude)